Automorphisms of free groups have asymptotically periodic dynamics

Gilbert Levitt, Martin Lustig

AUTOMORPHISMS OF FREE GROUPS HAVE ASYMPTOTICALLY PERIODIC DYNAMICS

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Abstract. We show that every automorphism α of a free group F_k of finite rank k has *asymptotically periodic* dynamics on F_k and its boundary ∂F_k: there exists a positive power α^q such that every element of the compactum $F_k \cup \partial F_k$ converges to a fixed point under iteration of α^q.

Further results about the dynamics of α as well as an extension from F_k to word-hyperbolic groups are given in the later sections.

Contents

Introduction and statement of results

Let F_k denote the free group of rank $k \geq 2$. Conjugation i_u by an element $u \in F_k$ has very simple dynamics. If $g \in F_k$ commutes with u, then g is a fixed point of i_u. If g does not commute with u, then the length of $(i_u)^n(g) = u^n g u^{-n}$ tends to infinity as $n \to +\infty$, and $u^n g u^{-n}$ converges to the infinite word $u^\infty = \lim_{n\to+\infty} u^n$.

On the space of infinite words (which may be viewed as the boundary ∂F_k), the action of i_u is simply left-translation by u. It has North-South (loxodromic) dynamics: u^∞ is an attracting fixed point (a sink), $u^{-\infty}$ is a repelling fixed point (a source), and $\lim_{n\to\pm\infty} u^n X = u^{\pm\infty}$ for every infinite word $X \neq u^{\pm\infty}$. Similar considerations apply to conjugation by any element of infinite order in a word hyperbolic group.

We proved in [24] that "most automorphisms" (in a precise sense) of a given hyperbolic group (e.g. F_k) have North-South dynamics on the boundary of the group. But, of course, interesting automorphisms usually are not "generic".

For instance, Nielsen studied mapping classes of surfaces by lifting them to the universal covering, and considering the action of various lifts on the circle at infinity S_∞ (the boundary of the surface group). He used lifts with more than two periodic points (see [30]), and one of his key results is that a lift f always has periodic points on S_∞ (equivalently, its rotation number on S_∞ is rational).

Since f induces a homeomorphism ∂f of the circle S_∞, this obviously implies that, for any X on the circle, the set of limit points of the sequence $\partial f^n(X)$ as $n \to +\infty$ is a periodic orbit of ∂f: we will say that ∂f has *asymptotically periodic dynamics*.

Our main results may be viewed as a generalization of these facts to arbitrary automorphisms of free (or hyperbolic) groups.

Let α be an automorphism of F_k. It induces canonically a homeomorphism $\partial\alpha$ on the boundary ∂F_k. The latter, a Cantor set, can be identified with the set of reduced right-infinite words in an arbitrary basis of F_k, or with the space of ends of any simplicial tree on which F_k acts freely. As usual, we provide F_k with the discrete topology and use ∂F_k to compactify F_k, thus obtaining $\overline{F}_k = F_k \cup \partial F_k$. One obtains from α and $\partial\alpha$ together a homeomorphism $\overline{\alpha} : \overline{F}_k \to \overline{F}_k$ of this compactum. This paper studies the dynamics of this homeomorphism. Let us first recall a few known results.

Theorem I [15, 25, 27]. *Let* $\alpha \in \mathrm{Aut}\,(F_k)$.

(1) *Every periodic orbit of* $\overline{\alpha}$ *has order bounded by* M_k, *where* M_k *depends only on* k, *and* $M_k \sim (k \log(k))^{1/2}$ *as* $k \to \infty$.
(2) $\partial\alpha$ *has at least two periodic points of period* $\leq 2k$.
(3) *A fixed point of* $\partial\alpha$ *which does not belong to the boundary of the fixed subgroup* $\mathrm{Fix}\,\alpha$ *is either attracting or repelling (sink or source). The number* $a(\alpha)$ *of orbits of the action of* $\mathrm{Fix}\,\alpha$ *on the set of attracting fixed points satisfies the "index inequality"* $\mathrm{rk}\,\mathrm{Fix}\,\alpha + \frac{1}{2}a(\alpha) \leq k$.
(4) *Every attracting fixed point of* $\partial\alpha$ *is superattracting with respect to the canonical Hölder structure of* ∂F_k, *with attraction rate* $\lambda_i \geq 1$. *If* $\lambda_i > 1$, *then* λ_i *is the exponential growth rate of some conjugacy class under iteration of* α. *There are at most* $\frac{3k-2}{4}$ *distinct such growth rates.*

An element $X \in \overline{F}_k$ is called *asymptotically periodic* (with respect to $\overline{\alpha}$) if the set $\omega(X)$ of accumulation points of the sequence $\overline{\alpha}^n(X)$ as $n \to +\infty$ is finite. This implies that $\omega(X)$ is a periodic orbit of $\overline{\alpha}$.

The automorphism $\alpha \in \mathrm{Aut}\,(F_k)$ has *asymptotically periodic dynamics* (on $\overline{F}_k$) if every $X \in \overline{F}_k$ is asymptotically periodic. Equivalently, α has asymptotically periodic dynamics if and only if there exists $q \geq 1$ such that, for every $X \in \overline{F}_k$, the sequence $\overline{\alpha}^{qn}(X)$ converges (see § 1.b). Our main result can now be stated as follows:

Theorem II. *Every automorphism* $\alpha \in \mathrm{Aut}\,(F_k)$ *has asymptotically periodic dynamics on* $\overline{F}_k$.

In particular, if $g \in F_k$ is not α-periodic, then the set of limit points of the sequence $\alpha^n(g)$ as $n \to +\infty$ is a periodic orbit of $\partial\alpha$. In other words, there exists q such that, for any N, the sequence consisting of the initial segment of length N of $\alpha^n(g)$ is eventually periodic with period q (the period q may be bounded by M_k, independently of g or α).

As an illustration, define an automorphism on the free group of rank 3 by $\alpha(a) = cb$, $\alpha(b) = a$, $\alpha(c) = ba$. Applying powers of α to a gives $a \mapsto cb \mapsto baa \mapsto acbcb \mapsto cbbaabaa \mapsto baaacbcbacbcb \mapsto \dots$, showing that $\alpha^n(a)$ limits onto an orbit of period 3. On the other hand $a^{-1} \mapsto b^{-1}c^{-1} \mapsto a^{-1}a^{-1}b^{-1} \mapsto b^{-1}c^{-1}b^{-1}c^{-1}a^{-1} \mapsto \dots$, and $\alpha^n(a^{-1})$ limits onto an orbit of period 2.

Other examples will be given in section 11. In particular, it is quite common for a boundary point of an α-invariant free factor F to be the limit of orbits well outside $\overline{F}$.

For an orientation-preserving homeomorphism of the circle, all periodic points have the same period, and existence of a periodic point is enough to imply asymptotically periodic dynamics. For automorphisms of free groups, there may be periodic points with different periods on the boundary (as shown by the above example). It is relatively easy to prove that periodic points exist, but much harder to prove that all points of the boundary are asymptotically periodic.

The basic tool in our approach is an *α-invariant* $\mathbf{R}$-tree, i.e. an $\mathbf{R}$-tree T with an action of F_k by isometries which is minimal, non-trivial, with *trivial arc stabilizers*, and α-invariant: its length function ℓ satisfies $\ell \circ \alpha = \lambda \alpha$ for some $\lambda \geq 1$. The automorphism α is then realized on T, in the sense that there is a homothety $H : T \to T$, with stretching factor $\lambda \geq 1$, such that $\alpha(w)H = Hw : T \to T$ for all $w \in F_k$. If $\lambda = 1$, the tree T is simplicial and H is an isometry. If $\lambda > 1$, the action of F_k on T is non-discrete; in fact, every F_k-orbit is dense in T.

In most cases, the map H has a fixed point Q (in T or in its metric completion $\overline{T}$). The stabilizer of Q is an α-invariant subgroup $\mathrm{Stab}\, Q \subset F_k$, which has rank strictly smaller than k [16]. This allows us to set up the proof of our main result as a proof by induction over the rank k.

For $g \in F_k$, we study the behavior of the sequence $\alpha^n(g)$ through that of the sequence $\alpha^n(g)Q = H^n(gQ)$, where Q is a fixed point of H.

There are three main cases (see § 3).

- If $gQ = Q$, we use the induction hypothesis.

- If $\lambda > 1$ and $H^n(gQ)$ goes out to infinity in T in a definite direction, then $\alpha^n(g)$ converges to an attracting fixed point of $\partial\alpha$.

- If $H^n(gQ)$ "turns around Q", one shows that $\alpha^n(g)$ accumulates onto a periodic orbit contained in $\partial\mathrm{Stab}\, Q$, using a cancellation argument given in § 2.

Similar arguments (given in § 4) make it possible to understand the behavior of $\partial\alpha^n(X)$, for $X \in \partial F_k$, when there are simplicial invariant trees, in particular when α is a polynomially growing automorphism.

The general case is dealt with in §§ 5 through 8. It makes use of the point $Q(X)$ introduced in [26], which reflects the dynamics of $\partial\alpha$ on X and allows us to extend the approach from the special cases dealt with previously. The proof consists of geometric arguments on relative train tracks, and involves the asymptotic behavior of four distinct ways of measuring length under iteration of α (and of α^{-1}). A sketch of the proof will be given at the beginning of § 5.

In § 9, we prove a few more results about dynamics of automorphisms of free groups. We show that F_k acts discretely on a suitable product of trees (a result proved in [2] and [28] for irreducible automorphisms). We study the bipartite graph whose vertices are the attracting and repelling fixed points of $\partial\alpha$, for α irreducible. We show that, for an arbitrary α, the number of different periods appearing in the dynamics of $\overline{\alpha}$ is bounded by a number depending only on k and growing roughly like $e^{\sqrt{k}}$. We also give a short proof of a result of [5] constructing automorphisms with many fixed points.

In § 10, we explain how to adapt the arguments of §§ 2, 3, 4 to hyperbolic groups. Our main result is:

Theorem III. *Let $\alpha \in \mathrm{Aut}(\Gamma)$, with Γ a virtually torsion-free hyperbolic group.*

(1) *Periodic orbits of $\overline{\alpha}$ have at most M points, with M depending only on Γ.*
(2) *Every $g \in \Gamma$ is asymptotically periodic.*
(3) *If Γ is one-ended, or α is polynomially growing, then every $X \in \partial\Gamma$ is asymptotically periodic.*

It is not known whether all hyperbolic group are virtually torsion-free (see [22]). In any case, it seems reasonable to conjecture that all automorphisms of hyperbolic groups have asymptotically periodic dynamics.

We start § 10 by giving a proof of an unpublished result by Shor [35]: Up to isomorphism, there are only finitely many fixed subgroups in a given torsion-free hyperbolic group. As in [35], we use results by Sela [34], Guirardel [17], Collins-Turner [10], but we simplify the proof by using the fact (proved in [23]) that $\mathrm{Aut}(\Gamma)$ contains only finitely many torsion conjugacy classes, for Γ a torsion-free hyperbolic group.

Theorem III is first proved for torsion-free groups. For one-ended groups, we use the simplicial tree (with cyclic edge stabilizers) given by the JSJ splitting. For free products, we use an $\mathbf{R}$-tree constructed from the train tracks of [10]. Finally, we extend our results to virtually torsion-free groups.

We conclude the paper by a section devoted to examples and questions.

1. Preliminaries

1.a. F_k and its boundary.

Let F_k be a free group of rank $k \geq 2$. Its boundary (or space of ends) ∂F_k is a Cantor set, upon which F_k acts by left translations. It compactifies F_k into $\overline{F}_k = F_k \cup \partial F_k$. The boundary ∂J of a finitely generated subgroup $J \subset F_k$ embeds naturally into ∂F_k. If $g \in F_k$ is nontrivial, we let $g^{\pm\infty} \in \partial F_k$ be the limit of g^n as $n \to \pm\infty$.

If we choose a free basis $\mathcal{A}$ of F_k, we may view ∂F_k as the set of right-infinite reduced words. The Gromov scalar product $(X|Y)$ of two elements $X, Y \in \overline{F}_k$ is the length of their maximal common initial subword. A sequence X_n in $\overline{F}_k$ converges to $X \in \partial F_k$ if and only if $(X_n|X) \to \infty$.

An automorphism $\alpha \in \mathrm{Aut}(F_k)$ is a quasi-isometry of F_k. It induces a homeomorphism $\partial\alpha : \partial F_k \to \partial F_k$, and also a homeomorphism $\overline{\alpha} = \alpha \cup \partial\alpha$ of the compact space $\overline{F}_k$. The conjugation $g \mapsto wgw^{-1}$ will be denoted by i_w. Note that ∂i_w is left-translation by w.

1.b. Limit sets and asymptotic periodicity.

Let f be a homeomorphism of a compact space K (for instance ∂F_k or $\overline{F}_k$). A point $x \in K$ is *periodic*, with period $q \geq 1$, if $f^q(x) = x$ and q is the smallest positive integer with this property. The set $\{x, f(x), \ldots, f^{q-1}(x)\}$ is a *periodic orbit* of order q. Given $y \in K$, the *ω-limit set* $\omega(y, f)$, or simply $\omega(y)$, is the set of limit points of the sequence $f^n(y)$ as $n \to +\infty$. It is compact, and invariant under f and f^{-1}. We observe:

Lemma 1.1. *Let f be a homeomorphism of a compact space K. Given $y \in K$ and $q \geq 1$, the following conditions are equivalent:*

1. *$\omega(y)$ is finite, and has q elements.*
2. *$\omega(y)$ is a periodic orbit of order q.*
3. *The sequence $f^{qn}(y)$ converges as $n \to +\infty$, and q is minimal for this property.*

Given $p \geq 2$, the set $\omega(y, f^p)$ is finite if and only if $\omega(y, f)$ is finite. □

If these equivalent conditions hold, we say that y is *asymptotically periodic.* In particular, we have defined asymptotically periodic elements of $\overline{F}_k$ (with respect to $\overline{\alpha}$).

We say that $\alpha \in \mathrm{Aut}\,(F_k)$ has *asymptotically periodic dynamics* (on $\overline{F}_k$) if every $X \in \overline{F}_k$ is asymptotically periodic. By assertion (1) of Theorem I, this is equivalent to the existence of $q \geq 1$ such that, for every $X \in \overline{F}_k$, the sequence $\overline{\alpha}^{qn}(X)$ converges as $n \to +\infty$.

1.c. Invariant trees.

As in [25], we will use as our basic tool an α-invariant $\mathbf{R}$-tree T with trivial arc stabilizers. We summarize its main properties.

Theorem 1.2 [15, 16]. *Given $\alpha \in \mathrm{Aut}\,(F_k)$, there exists an $\mathbf{R}$-tree T such that:*

(a) *F_k acts on T isometrically, non-trivially, minimally, with trivial arc stabilizers.*

(b) *There exist $\lambda \geq 1$ and a homothety H: $T \to T$ with stretching factor λ such that*

$$\alpha(g)H = Hg$$

for all $g \in F_k$ (viewing elements of F_k as isometries of T).

(c) *If $\lambda = 1$, then T may (and always will) be assumed to be simplicial (whereas all F_k-orbits are dense when $\lambda > 1$).*

(d) *Given $Q \in T$, its stabilizer* $\mathrm{Stab}\, Q$ *has rank $\leq k-1$, and the action of* $\mathrm{Stab}\, Q$ *on $\pi_0(T \setminus \{Q\})$ has at most $2k$ orbits. The number of F_k-orbits of branch points of T is at most $2k-2$.* □

A tree with these properties will simply be called an *α-invariant* $\mathbf{R}$*-tree* (with trivial arc stabilizers). Its length function is denoted by $\ell : F_k \to [0, \infty)$, it satisfies $\ell \circ \alpha = \lambda \ell$. An element $g \in F_k$ is *elliptic* if $\ell(g) = 0$ (i.e. if g has a fixed point), *hyperbolic* if $\ell(g) > 0$.

If a point $Q \in T$ with nontrivial stabilizer is fixed by H, the subgroup $\mathrm{Stab}\, Q$ is α-invariant. Since it has rank less that k, and $\partial \mathrm{Stab}\, Q$ embeds into ∂F_k, this will allow us to use induction on k.

A *ray* is (the image of) an isometric map ρ from $[0, \infty)$ or $(0, \infty)$ to T. It is an *eigenray* of H if $\rho(\lambda t) = H\rho(t)$, a *periodic ray* if it is an eigenray of some power of H. As usual, the *boundary* ∂T is the set of equivalence classes of rays. The action of F_k, and the map H, extend to ∂T.

We also consider the action of F_k on the metric completion $\overline{T}$ of T (when $\lambda = 1$, the tree T is simplicial, so $\overline{T} = T$). Note that points of $\overline{T} \setminus T$ have trivial stabilizer. Suppose $\lambda > 1$. The homothety H has a canonical extension to $\overline{T}$, with a unique fixed point $Q \in \overline{T}$. All eigenrays have origin Q. If a component of $\overline{T} \setminus \{Q\}$ is fixed by H (in particular if $Q \notin T$), that component contains a unique eigenray.

1.d. Bounded backtracking and $Q(X)$.

Let T be an α-invariant $\mathbf{R}$-tree with trivial arc stabilizers. Fix $Q \in \overline{T}$. When $\lambda > 1$, we always choose Q to be the fixed point of H.

Let Z be a geodesic metric space. We say that a map $f : Z \to \overline{T}$ has *bounded backtracking* if there exists $C > 0$ such that the image of any geodesic segment $[P, P']$ is contained in the C-neighborhood of the segment $[f(P), f(P')] \subset \overline{T}$. The smallest such C is the BBT-constant of f, denoted $BBT(f)$.

When Z is a simplicial tree, we only consider maps which are linear on each edge. If Z is a simplicial tree with a minimal free action of F_k, every F_k-equivariant $f : Z \to \overline{T}$ has bounded backtracking (see [2], [12], [15]).

In particular, let $Z_{\mathcal{A}}$ be the Cayley graph of F_k relative to a free basis $\mathcal{A} = \{a_1, \dots, a_k\}$. The map $f_{\mathcal{A}} : Z_{\mathcal{A}} \to \overline{T}$ sending the vertex g to gQ has bounded backtracking, with $BBT(f_{\mathcal{A}}) \leq \sum_{i=1}^k d(Q, a_i Q)$.

If $w, w' \in F_k$ and v is their longest common initial subword (in the basis $\mathcal{A}$), then vQ is $BBT(f_{\mathcal{A}})$-close to the segment $[wQ, w'Q]$ (property BBT2 of [15]). This is often used as follows: if $[Q, wQ] \cap [Q, w'Q]$ is long, then vQ is far from Q and therefore v is long.

Let ρ be a ray in T. By [15, Lemma 3.4], there is a unique $X = j(\rho) \in \partial F_k$ with the property that a sequence $w_n \in F_k$ converges to X if and only if the projection of $w_n Q$ onto ρ goes off to infinity. The map j is an F_k-equivariant injection from ∂T to ∂F_k satisfying $\partial\alpha \circ j = j \circ H$. If ρ is an eigenray, then $j(\rho)$ is a fixed point of $\partial\alpha$. When $\lambda > 1$, every fixed point of $\partial\alpha$ in $j(\partial T)$ is the image of an eigenray ρ.

The rest of this section will not be needed until § 5.

We suppose $\lambda > 1$. Then orbits are dense in T (see [33, Proposition 3.10]), and because $d(Q, \alpha^{-1}(a_i)Q) = \lambda^{-1} d(Q, a_i Q)$ there exist bases $\mathcal{A}$ with $BBT(f_{\mathcal{A}})$ arbitrarily small (this is a special case of [26, Corollary 2.3]).

In [26], we have associated a point $Q(X) \in \overline{T} \cup \partial T$ to every $X \in \partial F_k$. It may be thought of as the limit of $g_p Q$ as $g_p \to X$. Here we shall mostly be concerned with whether $Q(X)$ equals the fixed point Q or not, and we will work with the following alternative definition of $Q(X)$.

Given $X \in \partial F_k$, consider the set B_X consisting of points $R \in T$ which belong to the segment $[Q, wQ]$ for all $w \in F_k$ close enough to X (in other words, $R \in B_X$ if and only if there exists a neighborhood V of X in $\overline{F}_k$ such that $R \in [Q, wQ]$ for all $w \in V \cap F_k$). It is a connected subtree containing no tripod, and there are three possibilities.

If $B_X = \{Q\}$, we define $Q(X) = Q$.

If B_X is unbounded, it is an infinite ray ρ with origin Q and $j(\rho) = X$. We define $Q(X)$ as the point of ∂T represented by ρ.

The remaining possibility is that B_X is a closed or half-closed segment with origin Q. We then define $Q(X)$ as the other endpoint of this segment in $\overline{T}$ (it may happen that $Q(X) \notin B_X$).

It is easy to check that this definition of $Q(X)$ coincides with that of [26]. This implies that the assignment $X \mapsto Q(X)$ is F_k-equivariant. Note that in all cases $Q(\partial\alpha(X)) = H(Q(X))$. In particular, if $Q(X) = Q$, then $Q(\partial\alpha^n(X)) = Q$ for every $n \in \mathbf{Z}$.

Lemma 1.3. *Let T be an α-invariant $\mathbf{R}$-tree as in Theorem 1.2, with $\lambda > 1$. Let $Q \in \overline{T}$ be the fixed point of H. Let $\mathcal{A}$ be a basis of F_k, and let $f_{\mathcal{A}} : Z_{\mathcal{A}} \to \overline{T}$ be as above (sending g to gQ). Given $X \in \partial F_k$, let X_i be its initial segment of length i in the basis $\mathcal{A}$.*

1. *$d(X_iQ, B_X) \leq BBT(f_{\mathcal{A}})$ for all i.*
2. *If $Q(X) \in \overline{T}$, then $d(X_iQ, Q(X)) \leq 2BBT(f_{\mathcal{A}})$ for i large enough.*
3. *If $Q(X) \in \partial T$, the projection of X_iQ onto the ray B_X goes to infinity as $i \to \infty$.*

Proof. Suppose $d(X_iQ, Q) > BBT(f_{\mathcal{A}})$. The point located on the segment $[Q, X_iQ]$ at distance $BBT(f_{\mathcal{A}})$ from X_iQ belongs to $[Q, wQ]$ provided w starts with X_i, so is in B_X.

Suppose $Q(X) \in \overline{T}$. Fix $\varepsilon > 0$. For i large, the point $Q(X)$ is ε-close to $[Q, X_iQ]$, and therefore $d(X_iQ, Q(X)) \leq d(X_iQ, B_X) + \varepsilon$. Assertion (2) then follows from (1). Assertion (3) is clear. □

2. A lemma on asymptotic periodicity

Given $\alpha \in \mathrm{Aut}\,(F_k)$ and $w \in F_k$, we define $w_p = \alpha^{p-1}(w) \dots \alpha(w)w$ for $p \geq 1$ (with $w_1 = w$). Note that $w_r = w_s$ implies $w_{|r-s|} = 1$, and that $w_p = 1$ implies $\alpha^p(w) = \big(\alpha^{p-1}(w) \dots \alpha(w)\big)^{-1} = w$.

Recall that $X_n \in \overline{F}_k$ converges to $X \in \partial F_k$ if and only if $(X_n|X) \to \infty$. The relation $(X_n|Y_n) \to \infty$, between sequences in $\overline{F}_k$, is transitive (and does not depend on the basis $\mathcal{A}$).

Lemma 2.1. *Let $\alpha \in \mathrm{Aut}\,(F_k)$ and $w \in F_k$. Assume that for every $p \geq 1$ the elements w_p and w_p^{-1} are nontrivial and asymptotically periodic. Then any $x \in \overline{F}_k$ such that*

$$\lim_{n\to+\infty} \big(\overline{\alpha}^n(wx)|\overline{\alpha}^{n+1}(x)\big) = +\infty$$

is asymptotically periodic.

Proof. First suppose that w is not α-periodic. Then there exist $q \geq 1$ and $X, Y \in \partial F_k$ such that $\alpha^{qn}(w) \to X$ and $\alpha^{qn}(w^{-1}) \to Y$ as $n \to +\infty$. Write $\overline{\alpha}^{qn}(wx) = \alpha^{qn}(w)\overline{\alpha}^{qn}(x)$. Since w is not periodic, $\alpha^{qn}(w)$ gets long as $n \to \infty$, and therefore the maximum of $\big(\alpha^{qn}(w)|\overline{\alpha}^{qn}(wx)\big)$ and $\big(\alpha^{qn}(w^{-1})|\overline{\alpha}^{qn}(x)\big)$ goes to infinity with n. This implies that the maximum of $\big(X|\overline{\alpha}^{qn+1}(x)\big)$ and $\big(Y|\overline{\alpha}^{qn}(x)\big)$ goes to infinity.

It follows that the limit set of the sequence $\overline{\alpha}^{qn}(x)$ is contained in $\{\partial\alpha^{-1}(X), Y\}$. Thus x is asymptotically periodic.

Now suppose $\alpha(w) = w$. Then $\big(w\overline{\alpha}^n(x)|\overline{\alpha}^{n+1}(x)\big)$ goes to infinity with n. Fix an integer N. An easy induction shows $\lim_{n\to+\infty}\big(w^N\overline{\alpha}^n(x)|\overline{\alpha}^{n+N}(x)\big) = +\infty$. As above, we deduce that for n large at least one of $\big(w^N|\overline{\alpha}^{n+N}(x)\big)$, $\big(w^{-N}|\overline{\alpha}^n(x)\big)$ is large. It follows that the limit set of $\overline{\alpha}^n(x)$ is contained in $\{w^{-\infty}, w^{\infty}\}$.

The last case is when w is α-periodic with period $p \geq 2$. Then $\big(w\overline{\alpha}^{pn}(x)|\overline{\alpha}^{pn+1}(x)\big)$ goes to infinity with n, and our definition of w_p guarantees $\big(w_p\overline{\alpha}^{pn}(x)|\overline{\alpha}^{p(n+1)}(x)\big) \to \infty$. Since $\alpha^p(w_p) = w_p$, we reduce to the previous case (replacing α by α^p). $\square$

Remark 2.2. Suppose w belongs to a finitely generated α-invariant subgroup $J \subset F_k$. If x is as in Lemma 2.1, we have $\omega(x) \subset \partial J$ (because all points $X, Y, w^{\pm\infty}$ used in the proof belong to ∂J).

3. Limit sets of interior points

We first show:

Theorem 3.1. *Let $\alpha \in \operatorname{Aut}(F_k)$. If $g \in F_k$ is not α-periodic, the set of limit points of the sequence $\alpha^n(g)$ as $n \to +\infty$ is a periodic orbit of $\partial\alpha$.*

Proof. We consider an α-invariant $\mathbf{R}$-tree T as in Theorem 1.2. Because of Lemma 1.1, we are free to replace α by a positive power α^r whenever convenient. This has the effect of replacing H by H^r.

The proof is by induction on k. If $g \in \operatorname{Stab} Q$, with Q a fixed point of H, we may use the induction hypothesis (recall that $\operatorname{Stab} Q$ is α-invariant and has rank $< k$).

We distinguish two cases.

• First suppose $\lambda > 1$. Let $Q \in \overline{T}$ be the fixed point of H. If $gQ = Q$, we use induction on k. If $gQ \neq Q$, and the component $\mathcal{C}$ of $\overline{T} \setminus \{Q\}$ containing gQ is fixed by H (in particular if $Q \notin T$), that component contains an eigenray ρ (see § 1.c). Writing $\alpha^n(g)Q = \alpha^n(g)H^nQ = H^ngQ$, we see that $\alpha^n(g)$ converges to $j(\rho)$, a fixed point of $\partial\alpha$ (see § 1.d). Similarly, $\alpha^n(g)$ accumulates onto a periodic orbit of $\partial\alpha$ if the component $\mathcal{C}$ is H-periodic.

Assume therefore that $gQ \neq Q$ and $\mathcal{C}$ is not H-periodic. We will apply Lemma 2.1. Recall that the action of $\operatorname{Stab} Q$ on $\pi_0(T \setminus \{Q\})$ has finitely many orbits. Replacing α by a power, we may assume that there exists $w \in \operatorname{Stab} Q$ with $w\mathcal{C} = H\mathcal{C}$. Define $w_p = \alpha^{p-1}(w) \dots \alpha(w)w$ as in § 2. The elements w_p are all nontrivial, because w_p takes $\mathcal{C}$ onto $H^p\mathcal{C}$ (as checked by induction on p, using the equation $\alpha(w)H = Hw$). Using induction on k, we may assume that $w_p^{\pm 1}$ is asymptotically periodic.

Now we argue as in [15, p. 439]. The segments $[Q, wgQ]$ and $[Q, HgQ]$ intersect along a nondegenerate segment. Because $\lambda > 1$, the segments $[H^nQ, H^nwgQ] = [Q, \alpha^n(wg)Q]$ and $[H^nQ, H^{n+1}gQ] = [Q, \alpha^{n+1}(g)Q]$ intersect along a segment whose length goes to infinity with n. By bounded backtracking (property BBT2 of [15], see § 1.d), this implies that the scalar product $\big(\alpha^n(wg)|\alpha^{n+1}(g)\big)$ goes to infinity with n. Lemma 2.1 now concludes the proof (with $\omega(g) \subset \partial \mathrm{Stab}\, Q$ by Remark 2.2).

• When $\lambda = 1$, we view g and H as isometries of the simplicial tree T. Note that g has at most one fixed point (because arc stabilizers are trivial). If g is hyperbolic (i.e. it has no fixed point), its translation axis has compact intersection with the axis of H (if H is hyperbolic), or with the set of periodic points of H (if H is elliptic). Otherwise g would commute with some power H^r on a nondegenerate segment, implying $\alpha^r(g) = g$.

First suppose that H is hyperbolic, with axis A. Orient A by the action of H, and consider its two ends A^-, A^+. Choose $Q \in A$. As n goes to infinity, the projection of $gH^{-n}Q$ onto A remains far from A^- (because g does not fix A^-). It follows that the projection of $\alpha^n(g)Q = H^n gH^{-n}Q$ onto A goes off to A^+, and $\alpha^n(g)$ converges to the fixed point $j(A^+)$ of $\partial\alpha$.

If H is elliptic, let $\mathcal{P}$ be the subtree consisting of all H-periodic points. There exists a (unique) point Q of $\mathcal{P}$ such that $[Q, gQ] \cap \mathcal{P} = \{Q\}$ (it is the point of $\mathcal{P}$ closest to the fixed point of g if g is elliptic, to the positive end of the axis of g if g is hyperbolic). Replacing α by a power, we may assume $HQ = Q$.

If $gQ = Q$, we use induction on k. If not, we let $\mathcal{C}$ be the component of $T \setminus \{Q\}$ containing gQ, and we find $w \in \mathrm{Stab}\, Q$ with $w\mathcal{C} = H\mathcal{C}$, as in the proof when $\lambda > 1$. The components $\mathcal{C}_n = H^n\mathcal{C} = w_n\mathcal{C}$ are all distinct, because Q was chosen so that the germ of $[Q, gQ]$ at Q is not H-periodic. Since both points $\alpha^n(wg)Q = H^n wgQ$ and $\alpha^{n+1}(g)Q = H^{n+1}gQ$ belong to $\mathcal{C}_{n+1}$, the scalar product $\big(\alpha^n(wg)|\alpha^{n+1}(g)\big)$ goes to infinity and Lemma 2.1 applies. □

The argument that concludes this proof will be used again. We may state it as follows.

Lemma 3.2. *Let T be a minimal simplicial F_k-tree with trivial edge stabilizers. Given $Q \in T$ and distinct components $\mathcal{C}_n$ of $T \setminus \{Q\}$, there exists a sequence of numbers $m_n \to \infty$ such that, if a_n, b_n are elements of F_k with a_nQ and b_nQ both belonging to $\mathcal{C}_n$, then $(a_n|b_n) \geq m_n$.* □

From Theorem 3.1 we deduce:

Corollary 3.3. *Let $\alpha \in \mathrm{Aut}\,(F_k)$ and $w \in F_k$. If the sequence $w_n = \alpha^{n-1}(w)\ldots\alpha(w)w$ is not periodic (in particular if w is not α-periodic), its limit set is a periodic orbit of $\partial\alpha$.*

Proof. Extend α to $\beta \in \mathrm{Aut}\,(F_k * \mathbf{Z})$ by sending a generator t of $\mathbf{Z}$ to wt. Then $\beta^n(t) = w_n t$. The map $n \mapsto w_n$ is injective, because otherwise w_n would be periodic. Thus the length of w_n goes to infinity, implying that the sequences w_n and $\beta^n(t)$ have the same limit set in $\partial(F_k * \mathbf{Z})$. That set is contained in ∂F_k, and is a periodic orbit of $\partial\beta$ by Theorem 3.1. It is therefore a periodic orbit of $\partial\alpha$. □

The following observation will be useful in § 9.

Proposition 3.4. *Let T be an α-invariant $\mathbf{R}$-tree. Suppose that $\lambda > 1$ and the fixed point $Q \in \overline{T}$ of H has trivial stabilizer. Then α has no nontrivial periodic* *element in F_k. There is a bijection τ from $\pi_0(\overline{T} \setminus \{Q\})$ to the set of attracting periodic points of $\partial\alpha$, with $\tau \circ H = \partial\alpha \circ \tau$.*

Proof. If $\alpha^q(g) = g$, then $H^q gQ = \alpha^q(g)H^q Q = gQ$, so $g \in \operatorname{Stab} Q = \{1\}$. This proves the first assertion. By [15], $\partial\alpha$ has finitely many periodic points, each of them attracting or repelling.

Since $\operatorname{Stab} Q$ is trivial, $\overline{T} \setminus \{Q\}$ has finitely many components, each of them H-periodic. As seen in the proof of Theorem 3.1, the asymptotic behavior of a sequence $\alpha^n(g)$ depends only on the component $\mathcal{C}$ containing gQ. The attracting periodic point of $\partial\alpha$ associated to $\mathcal{C}$ is $\tau(\mathcal{C}) = j(\rho)$, where ρ is the H-periodic ray contained in $\mathcal{C}$. ☐

Remark 3.5. Suppose $\lambda > 1$, but don't assume that $\operatorname{Stab} Q$ is trivial. The proof of Theorem 3.1 shows that either $\omega(g) \subset \overline{\operatorname{Stab} Q}$, or $\alpha^n(g)$ accumulates onto a periodic orbit of $\partial\alpha$ associated to H-periodic rays.

4. The simplicial case

This section is devoted to the proof of the following result.

Theorem 4.1. *Assume that Theorem II is true for free groups of rank $< k$. Let $\alpha \in \operatorname{Aut}(F_k)$. If there exists a simplicial α-invariant tree T as in Theorem 1.2, then every $X \in \partial F_k$ is asymptotically periodic for α.*

Corollary 4.2. *Theorem II is true for polynomially growing automorphisms of F_k.*

Proof of Corollary 4.2. It is proved by induction on k. Given $\alpha \in \operatorname{Aut}(F_k)$, consider T as in Theorem 1.2, with length function ℓ. Choose $w \in F_k$ with $\ell(w) > 0$. If α is polynomially growing, then $\ell(\alpha^n(w))$ has subexponential growth (as translation length is bounded from above by a constant times word length). Since $\ell(\alpha^n(w)) = \lambda^n \ell(w)$, we have $\lambda = 1$ and Theorem 4.1 applies. ☐

To prove Theorem 4.1, we argue as in the proof of Theorem 3.1 when $\lambda = 1$ (keeping the same notations). If the isometry H is hyperbolic, then $\partial\alpha^n(X)$ converges to the fixed point $j(A^+)$ for every $X \neq j(A^-)$, so we assume that H is elliptic.

First suppose that X belongs to $\partial\operatorname{Stab} R$ for a (unique) point $R \in T$. Let Q be the point of $\mathcal{P}$ closest to R. Replacing α by a power, we may assume $HQ = Q$. If $R = Q$ (i.e. if R is fixed by H), we use induction on k. If $R \neq Q$, we let $\mathcal{C}$ be the component of $T \setminus \{Q\}$ containing R and we choose $w \in \operatorname{Stab} Q$ with $w\mathcal{C} = H\mathcal{C}$ (replacing α by a power if needed).

Let g_p be a sequence in $\operatorname{Stab} R$ converging to X. Since $g_pQ \in \mathcal{C}$, we can apply Lemma 3.2 for each value of p, with $\mathcal{C}_n = H^n\mathcal{C}$, $a_n = \alpha^n(wg_p)$, $b_n = \alpha^{n+1}(g_p)$. We find $\big(\alpha^n(wg_p)|\alpha^{n+1}(g_p)\big) \geq m_n$, and therefore $\big(\partial\alpha^n(wX)|\partial\alpha^{n+1}(X)\big)$ tends to infinity as $n \to +\infty$. Lemma 2.1 concludes the proof, since w belongs to an α-invariant subgroup of rank $< k$.

If X does not belong to any $\partial \mathrm{Stab}\, R$, then, since T is simplicial and arc stabilizers are trivial, $X = j(e)$ for some end e of T. We may assume that e is not H-periodic. We apply Lemmas 3.2 and 2.1 as above, using the point $Q \in \mathcal{P}$ closest to e and the component $\mathcal{C}$ of $T \setminus \{Q\}$ containing e. The existence of this point Q is not obvious, however, because e could be a limit of periodic points of H whose period tends to infinity. But this is ruled out by the following fact, and the proof of Theorem 4.1 is complete.

Lemma 4.3. *Let α, T, H be as in Theorem 1.2. Assume $\lambda = 1$. There exists M (depending only on k) such that all periodic points of H have period less than M.*

Proof. If the action of F_k on T is free, some fixed power of H is hyperbolic or is the identity (because H lifts an automorphism of the finite graph T/F_k). We assume from now on that the action is not free.

Suppose that a vertex Q, with $\mathrm{Stab}\, Q$ nontrivial, is H-periodic with period q. Since $\mathrm{Stab}\, Q$ is invariant under α^q, we know that $\partial \mathrm{Stab}\, Q$ contains a periodic point X of $\partial\alpha^q$. Viewed as a periodic point of $\partial\alpha$, the point X has period divisible by q (because stabilizers of distinct vertices H^iQ, $1 \leq i \leq q$, have disjoint boundaries), and this forces $q \leq M_k$ by [25] (see Theorem I in the introduction).

We have now obtained a bound for periods (under H) of vertices with nontrivial stabilizer. Since the action of F_k on T is minimal and not free, any vertex Q with trivial stabilizer belongs to a segment $[Q_1, Q_2]$, with $\mathrm{Stab}\, Q_1$ and $\mathrm{Stab}\, Q_2$ nontrivial but $\mathrm{Stab}\, R$ trivial for every interior point R of $[Q_1, Q_2]$. If Q is H-periodic, so are Q_1 and Q_2 because vertices with trivial stabilizer have finite valence in T. We know how to bound the period of Q_1 and Q_2, and this leads to a bound for the period of Q. □

5. A reduction

From now on we will consider an α-invariant $\mathbf{R}$-tree T with $\lambda > 1$, as in Theorem 1.2. We always denote by Q the fixed point of H (in $\overline{T}$). We fix $X \in \partial F_k$, and we study the sequence $\partial\alpha^n(X)$.

For the reader's convenience, we now give a quick overview of §§ 5-8.

In § 3, we studied the sequence $\alpha^n(g)$ by applying powers of H to the point gQ. Instead of gQ, we now consider the point $Q(X)$ introduced in [26] (see § 1.d). It is either a point of the metric completion $\overline{T}$, or an end of T. It may be thought of as the limit of g_pQ as $g_p \to X$.

When $Q(X) \neq Q$, a naive argument works: the behavior of $\partial\alpha^n(X)$ is the same as that of $\alpha^n(g)$ for $g \in F_k$ close to X. If $X \in \partial \mathrm{Stab}\, Q$, we use induction on k. The hard case is when $Q(X) = Q$ but X does not belong to $\partial \mathrm{Stab}\, Q$. This happens when X is a repelling fixed point of $\partial\alpha$, and the task here is indeed to show that this is essentially the only possibility (Theorem 5.1).

The difficulty lies in the fact that cancellation within the infinite word X due to the contracting nature of the combinatorics of a repelling X can be mixed with cancellation in the initial subword of X coming from accumulations of finite elements from $\mathrm{Stab}\, Q$.

We consider an improved relative train track map $f_0 : G \to G$ representing α (in the sense of [3]), with an exponential top stratum. If there is an indivisible Nielsen path η meeting the top stratum, then for each lift $[a, b]$ of η in the universal covering $\widetilde{G}$ we create a new edge between a and b (this is similar to the addition of Nielsen faces in [29]). The resulting space $\widetilde{\mathcal{G}}$ is a cocompact F_k-space, but we also consider the non-proper space $\widetilde{\mathcal{G}}_{PF}$ obtained from $\widetilde{\mathcal{G}}$ by contracting all edges not in the top stratum (it resembles the coned-off Cayley graph of [13]). We show that, when $Q(X) = Q$, the point at infinity X behaves like a repelling point for the train track map acting on $\widetilde{\mathcal{G}}_{PF}$ (Lemma 8.2).

We also consider a train track map $f'_0 : G' \to G'$ representing α^{-1}, which is paired with f_0 in the sense of [3, § 3.2]. Because of the pairing, the spaces $\widetilde{\mathcal{G}}_{PF}$ and $\widetilde{\mathcal{G}}'_{PF}$ are quasi-isometric. It follows that X behaves like an attracting point in $\widetilde{\mathcal{G}}'_{PF}$, and there is a legal ray going out towards X in $\widetilde{\mathcal{G}}'$ (Lemma 8.4). We then conclude by applying these facts to all points $\partial\alpha^{-n}(X)$.

We now proceed to reduce Theorem II to the following result, whose proof will be completed in § 8.

Theorem 5.1. *Let $\alpha \in \mathrm{Aut}\,(F_k)$. If there is no simplicial α-invariant tree (with trivial arc stabilizers), there is an α-invariant $\mathbf{R}$-tree T with $\lambda > 1$ such that, for every $X \in \partial F_k$, at least one of the following holds:*

(1) $X \in \partial \mathrm{Stab}\, Q$.
(2) $Q(X) \neq Q$.
(3) *There exist $q \geq 1$ and $w \in \mathrm{Stab}\, Q$ such that X is a fixed point of $w\partial\alpha^q = \partial(i_w \circ \alpha^q)$.*

See § 1.d for the definition of $Q(X) \in \overline{T} \cup \partial T$, and recall that $i_w(g) = wgw^{-1}$.

Assuming this theorem, we prove Theorem II by induction on k. Consider $\alpha \in \mathrm{Aut}\,(F_k)$. If there is a simplicial α-invariant tree, we apply Theorem 4.1. If not, we fix $X \in \partial F_k$ and we consider the three possibilities of Theorem 5.1.

• If $X \in \partial \mathrm{Stab}\, Q$, we use the induction hypothesis (recall that $\mathrm{Stab}\, Q$ is α-invariant, with rank $< k$).

• If $Q(X) \neq Q$, let $\mathcal{C}$ be the component of $\overline{T} \setminus \{Q\}$ containing $B_X \setminus \{Q\}$. Choose a basis $\mathcal{A}$ of F_k with $BBT(f_{\mathcal{A}})$ small with respect to $d(Q(X), Q)$. By Lemma 1.3, we have $X_i Q \in \mathcal{C}$ for i large. Replacing α by a power if necessary, we may assume (as done in the previous sections) that there exists $w \in \mathrm{Stab}\, Q$ such that $H\mathcal{C} = w\mathcal{C}$. We argue as in the proof of Theorem 3.1, "uniformly". There are two cases.

If $w = 1$, let ρ be the eigenray of H contained in $\mathcal{C}$. The element $j(\rho) \in \partial F_k$ is a fixed point of $\partial\alpha$, and we show $\lim_{n \to +\infty} \partial\alpha^n(X) = j(\rho)$.

The distance from Q to the projection of $X_i Q$ onto ρ is greater than some $\varepsilon > 0$ for i large. It follows that the distance from Q to the projection of $\alpha^n(X_i)Q = H^n(X_i Q)$ onto ρ is greater than $\lambda^n \varepsilon$, independently of i. By bounded backtracking, as in the proof of Lemma 3.4 in [15], we obtain that $\big(j(\rho) | \alpha^n(X_i)\big)$ goes to infinity as $n \to +\infty$, uniformly in i. This shows $\lim_{n \to +\infty} \partial\alpha^n(X) = j(\rho)$, as required.

If $w \neq 1$, we observe that the length of $[Q, wX_iQ] \cap [Q, HX_iQ]$ is bounded away from 0, so that $\lim_{n\to+\infty}\big(\overline{\alpha}^n(wX_i)|\overline{\alpha}^{n+1}(X_i)\big) = +\infty$ uniformly in i. Therefore $\big(\overline{\alpha}^n(wX)|\overline{\alpha}^{n+1}(X)\big)$ goes to infinity and Lemma 2.1 applies.

• Replacing α by α^q, we may assume that X is a fixed point of $\partial\beta$, with $\beta = i_w \circ \alpha$. Write $\partial\alpha^n(X) = w_n^- X$ with $w_n^- = \alpha^{n-1}(w^{-1}) \dots \alpha(w^{-1})w^{-1} \in \operatorname{Stab} Q$. If there exist $r \neq s$ with $w_r^- = w_s^-$, then X is $\partial\alpha$-periodic and we are done. We therefore assume $|w_n^-| \to \infty$. We may also assume that the cancellation in the product $w_n^- X$ is bounded, since otherwise $X \in \partial\operatorname{Stab} Q$. It follows that the sequence $\partial\alpha^n(X)$ has the same limit points as the sequence w_n^-, and we conclude by Corollary 3.3.

6. The train track and the tree

In this section we let the map $f_0 : G \to G$ be an improved relative train track representative for (a power of) α, in the sense of [3, 5.1.5]. Basic references are [6] and §§ 2.5, 5.1 of [3].

We assume that the top stratum G_t is exponentially growing. Edges in G_t are called top edges, edges in the other strata are called zero edges. An illegal turn in G is a turn between two top edges where folding occurs when some power of f_0 is applied (see [6]).

There is at most one indivisible Nielsen path (INP) η in G meeting the top stratum (see [3, 5.1.5 and 5.1.7] for its properties). It has precisely one illegal turn (its tip), and $f_0(\eta)$ is homotopic to η relative to its endpoints (which are fixed by f_0). If η is a loop, there is no cancellation in η^2 and the turn at the midpoint of η^2 is legal.

We start by changing G into a "train track with shortcut" $\mathcal{G}$, by adding a new zero edge if needed. This is a very special case of the "train tracks with Nielsen faces" introduced in [29].

Creating a shortcut.

If there is no INP η as above, we let $\mathcal{G} = G$. If there is one, we first subdivide G (if needed) to ensure that the endpoints of η are vertices. Then we enlarge G into $\mathcal{G}$ by adding a new zero edge e with the same endpoints as η (these endpoints are equal if the stratum is geometric). The map f_0, originally defined on G, extends to $\mathcal{G}$ (it is defined as the identity on e).

We fix a retraction $r_0 : \mathcal{G} \to G$ mapping e onto η in a locally injective way. It induces a homomorphism $(r_0)_* : \pi_1(\mathcal{G}) \to \pi_1(G) \simeq F_k$, and we let $\widetilde{\mathcal{G}}$ be the corresponding F_k-covering (it consists of the universal covering $\widetilde{G}$ of G, together with lifts of e; if a 2-cell is attached to $\mathcal{G}$ along $\eta \cup e$, then $\widetilde{\mathcal{G}}$ becomes the 1-skeleton of the universal covering).

We define top and zero edges on $\widetilde{\mathcal{G}}$ as on $\mathcal{G}$, we lift r_0 to a retraction r, and we define f to be the lift of f_0 to $\widetilde{\mathcal{G}}$ that satisfies $\alpha(w)f = fw$ for every deck transformation $w \in F_k$.

Legal paths.

We define an *illegal turn in $\mathcal{G}$* as either an illegal turn in G, or a turn between e and a top edge which becomes degenerate or illegal in G when e is replaced by η (recall that η starts and ends with segments contained in the top stratum G_t; if an endpoint x of η lies in the interior of a top edge of G, exactly one of the turns at the corresponding end of e is illegal). A turn in $\widetilde{\mathcal{G}}$ is illegal if it projects onto an illegal turn.

Paths will always be assumed to start and end at vertices. A *legal path*, in $\mathcal{G}$ or in $\widetilde{\mathcal{G}}$, is a path with no illegal turn. Note that r_0 maps a legal path locally injectively, and the image has illegal turns only at the tip of η. Given vertices $x, y \in \widetilde{\mathcal{G}}$, we let $ILT(x, y)$ be the minimum number of illegal turns on paths from x to y in $\widetilde{\mathcal{G}}$.

Now consider the action of f_0 on a legal path $\gamma \subset \mathcal{G}$. Write γ as a concatenation of copies of e and legal paths in G. When f_0 is applied, the paths in G remain legal and nontrivial (after reduction), by property (RTT-ii) of [6], and the turns at the endpoints of e remain legal. It follows that f_0 and f map legal paths to legal paths. In particular, we have $ILT(f(x), f(y)) \leq ILT(x, y)$.

Lemma 6.1 [3, 29]. *Let x, y be vertices of $\widetilde{\mathcal{G}}$. For p large enough, there is a legal path joining $f^p(x)$ to $f^p(y)$. More precisely, given $\delta > 0$, there exists p such that $ILT(f^p(x), f^p(y)) \leq \delta ILT(x, y)$ for all vertices $x, y \in \widetilde{\mathcal{G}}$.*

Proof. The first assertion is Lemma 3.2 of [29], or Lemma 4.2.6 of [3] (note that the definition of a legal path used here is slightly stronger than the one used in [29], but it is easy to make the path legal in the stronger sense).

By Remark 3.3 of [29], there exists p such that $ILT(f^p(x), f^p(y)) = 0$ if $ILT(x, y) = 1$. This easily implies $ILT(f^p(x), f^p(y)) \leq \frac{1}{2} ILT(x, y)$ for all x, y. The extension to an arbitrary δ is immediate. ☐

The PF-metric and the invariant tree.

Since the top stratum G_t is assumed to be exponentially growing, the transition matrix associated to G_t has a Perron-Frobenius eigenvalue $\lambda > 1$. Using components of an eigenvector, one assigns a PF-length $|E|_{PF}$ to each top edge $E \subset \mathcal{G}$, with the property that the total PF-length of $f_0(E)$ is $\lambda |E|_{PF}$. More generally, f_0 multiplies the PF-length of legal paths by λ.

On each of the spaces $\widetilde{G}$ and $\widetilde{\mathcal{G}}$, we define a *PF-pseudo-distance* d_{PF} by giving top edges their PF-length and assigning length 0 to the zero edges (as usual, the distance between two points is the length of the shortest path joining them). Note that these two distances differ on $\widetilde{G}$: the endpoints of a lift of η have PF-distance 0 in $\widetilde{\mathcal{G}}$ but not in $\widetilde{G}$. The associated metric spaces $\widetilde{G}_{PF}$ and $\widetilde{\mathcal{G}}_{PF}$ are geodesic (because there is a positive lower bound for the PF-length of top edges), but in general not proper.

We also define the *simplicial distance* d, for which top edges have their PF-length and zero edges have length 1. This makes $\widetilde{G}$ and $\widetilde{\mathcal{G}}$ into proper geodesic spaces, with cocompact actions of F_k. Note that $d_{PF} \leq d$.

Now recall the construction of an α-invariant $\mathbf{R}$-tree T with trivial arc stabilizers given in [15]. The pseudo-distance d_{PF} on the tree $\widetilde{G}$ satisfies $d_{PF}(f(x), f(y)) \leq \lambda d_{PF}(x, y)$, with equality if x and y are joined by a legal path. We define

$$d_\infty(x, y) = \lim_{p\to\infty} \lambda^{-p} d_{PF}(f^p(x), f^p(y))$$

and we let T be the metric space associated to this pseudo-distance. We denote by π the quotient map $\pi : \widetilde{G} \to T$. The map f induces a homothety $H : T \to T$, with $\pi \circ f = H \circ \pi$. It is proved in [15] that T and H satisfy the conditions of Theorem 1.2.

Note that the endpoints of a given lift of η have the same image in T, since their images by f also bound a lift of η. This implies that, if we apply the above construction to d_{PF} on $\widetilde{\mathcal{G}}$ rather than on $\widetilde{G}$, we get the same space T, and the quotient map $i : \widetilde{\mathcal{G}} \to T$ extends π.

The maps $i : \widetilde{\mathcal{G}} \to T$ and $\pi : \widetilde{G} \to T$ are F_k-equivariant, and 1-Lipschitz for the respective d_{PF} (hence also for the simplicial metrics). Also note that i and $\pi \circ r$ are uniformly close.

The map π, defined on the tree $\widetilde{G}$, sends legal paths PF-isometrically into T, and (see § 1.d) it has bounded backtracking (the image of a geodesic segment $[x, y]$ is close to $[\pi(x), \pi(y)]$). We show that $i : \widetilde{\mathcal{G}}_{PF} \to T$ has similar properties. This is a quite different statement, because geodesics in $\widetilde{\mathcal{G}}_{PF}$ can be very different from simplicial geodesics.

Lemma 6.2 [29, Lemma 3.9 and Proposition 4.2]. *A legal path $\gamma \subset \widetilde{\mathcal{G}}$ is PF-geodesic and maps to T PF-isometrically. The map $i : \widetilde{\mathcal{G}}_{PF} \to T$ has bounded backtracking.*

Proof. Let $x, y \in \widetilde{G}$ be the endpoints of γ. Let L be the PF-length of γ, and m the number of lifts of e contained in γ. Then $f^p(\gamma)$ is a legal path with PF-length $\lambda^p L$, containing only m lifts of e. Its image by r is embedded in $\widetilde{G}$ and has PF-length $\lambda^p L + m|\eta|_{PF}$. Letting $p \to \infty$ shows that $i(x)$ and $i(y)$ have distance L in T. Since i is 1-Lipschitz for the PF-metric, we see that γ is PF-geodesic and $i_{|\gamma}$ is isometric for the PF-metric.

To prove bounded backtracking, it suffices to consider a PF-geodesic $\gamma = [x, y]$ in $\widetilde{\mathcal{G}}$ with $i(x) = i(y)$, and to bound the diameter of $i(\gamma)$. We may assume $x, y \in \widetilde{G}$. Let δ be the geodesic $[x, y]$ in the tree $\widetilde{G}$. We know that π has bounded backtracking, so $\pi(\delta)$ is close to $i(x)$. Consider $z \in \gamma$ with $i(z)$ far from $i(x)$. Since i and $\pi \circ r$ are close, we have $r(z) \notin \delta$. Because $\widetilde{G}$ is a tree, z belongs to a subarc $[x', y'] \subset \gamma$ with $r(x') = r(y') \in \delta$. The points x' and y' are close in $\widetilde{\mathcal{G}}$ for the simplicial distance, hence also for the PF-distance. Since γ is PF-geodesic (and i is 1-Lipschitz for the PF-metric), $i(z)$ is close to $i(x')$, hence to $\pi(r(x'))$, hence to $i(x)$. $\square$

Elliptic elements.

A loop in G represents a conjugacy class in $\pi_1(G) \simeq F_k$. So does a loop in $\mathcal{G}$, through the homomorphism $(r_0)_* : \pi_1(\mathcal{G}) \to \pi_1(G) \simeq F_k$.

Lemma 6.3. *Given a conjugacy class $[u]$ in F_k, the following are equivalent:*

(1) *$[u]$ may be represented by a loop in the zero part of $\mathcal{G}$;*
(2) *$[u]$ may be represented by a loop in the zero part of G, or $[u]$ is α-invariant (i.e. $\alpha(u)$ is conjugate to u);*
(3) *$[u]$ is elliptic in T (i.e. u fixes a point in T).*

Proof. For $p > 0$, each of the three conditions holds for $[u]$ if and only if it holds for $\alpha^p([u])$ (for the first two conditions, use Scott's lemma [3, 6.0.6] to argue that $[u]$ may be represented by a loop in the zero part if $\alpha^p([u])$ does). By Lemma 6.1 we may assume that $[u]$ is represented by a legal loop γ in $\mathcal{G}$. By Lemma 6.2, the translation length $\ell(u)$ of $[u]$ in T is then the PF-length of γ.

If this length is positive, then u is hyperbolic in T. It cannot be represented by a loop in a zero part, or be α-invariant (since $\ell(\alpha(u)) = \lambda\ell(u)$). If the length is 0, then $[u]$ is elliptic and γ is contained in the zero part of $\mathcal{G}$. If $[u]$ cannot be represented by a loop in the zero part of G, then by [3, 6.0.2] u is (conjugate to) a power of η (which is a loop). It follows that $[u]$ is α-invariant. □

We may be more specific, working with elements of F_k rather than with conjugacy classes. Let Z_i be the non-contractible components of the zero part of G. Images of $\pi_1(Z_i)$ in F_k are free factors. By [3, 5.1.5], these factors are α-invariant (up to conjugacy). Not much changes when we pass from G to $\mathcal{G}$. If G_t is not geometric, zero components of $\mathcal{G}$ yield the same subgroups [3, 5.1.7]. In the geometric case, there is one more component, whose image in F_k is the cyclic group generated by the loop η (which runs once around the closed INP).

If Z is a non-contractible zero component of $\mathcal{G}$, and $\hat{Z}$ is a component of its preimage in $\widetilde{\mathcal{G}}$, the image of $\hat{Z}$ in T is a point $i(\hat{Z})$. By Lemma 6.3 the image of $\pi_1(Z)$ in F_k is the full stabilizer of $i(\hat{Z})$, and every non-trivial stabilizer arises in this way: non-trivial stabilizers of points of T are precisely conjugates of fundamental groups of non-contractible zero components of $\mathcal{G}$.

7. Geometry on the train track

Spaces quasi-isometric to trees.

If c is a continuous map from $[0, 1]$ to a tree, the image of c contains the geodesic segment between $c(0)$ and $c(1)$. Spaces quasi-isometric to a tree have a similar property.

Lemma 7.1. *Let Y be a geodesic metric space quasi-isometric to a tree. There exists C such that, if γ is a geodesic segment and γ' is any path with the same endpoints, then every $P \in \gamma$ is C-close to some $Q \in \gamma'$.*

Proof. Let $f : Y \to S$ be a quasi-isometry to a tree. Since $f(\gamma)$ is a quasi-geodesic, $f(P)$ is close to some R on the geodesic γ_0 joining the f-images of the endpoints of γ, and R is close to the image of some $Q \in \gamma'$. The distance from P to this point Q is bounded. □

This basic fact may be extended in several ways.

If the endpoints of γ' are only C-close to those of γ, then every $P \in \gamma$ at distance at least $2C$ from the endpoints is C-close to γ'. If points $P_1, P_2, \dots$ appear in this order on γ, each at distance at least $2C$ from the previous one, we may assume that the associated points Q_i appear in the same order on γ' (construct Q_i inductively).

Lemma 7.1 also holds if γ, γ' are (possibly infinite) quasi-geodesics with the same endpoints, with C depending only on the quasi-geodesy constants of γ.

K-PF-geodesics.

Let $\widetilde{\mathcal{G}}$ be as above. It is equipped with the simplicial distance d, the PF-distance d_{PF}, and the distance d_∞. Recall that $d_\infty \le d_{PF} \le d$. The space $(\widetilde{\mathcal{G}}, d)$ is proper and quasi-isometric to F_k, hence to a tree. We identify $\partial\widetilde{\mathcal{G}}$ with ∂F_k. We fix C as in Lemma 7.1.

The words "geodesic" and "quasi-geodesic" will always refer to the simplicial metric. We write K-quasi-geodesic instead of (K, K)-quasi-geodesic. A geodesic relative to d_{PF} will be called a PF-geodesic. If a PF-geodesic is also K-quasi-geodesic (with respect to d), we call it a K-PF-geodesic. Recall that legal paths are PF-geodesics.

Lemma 7.2. *If K is large enough, the following properties hold:*

1. *There exists a K-PF-geodesic between any two points of $\widetilde{\mathcal{G}} \cup \partial\widetilde{\mathcal{G}}$.*
2. *If there exists a legal path between two points of $\widetilde{\mathcal{G}}$, there exists a legal K-quasi-geodesic between them.*

Proof. Given a geodesic segment γ, first choose a PF-geodesic γ' with the same endpoints. Place points P_i, Q_i as above (after Lemma 7.1), with $d(P_i, P_{i+1}) = 2C$ (we assume that the points Q_i are vertices). Replace the segment of γ' between Q_i and Q_{i+1} by another PF-geodesic segment, with simplicial length as small as possible. The resulting curve is a PF-geodesic, and it is uniformly quasi-geodesic because the set of pairs (Q_i, Q_{i+1}) is finite up to deck transformations ($\widetilde{\mathcal{G}}$ is a locally finite graph). If γ is infinite, we apply the usual diagonal argument.

If there is a legal γ', we choose the replacement of $[Q_i, Q_{i+1}]$ among legal paths with the same initial and final edges (so that the turns at the Q_i's remain legal). $\square$

Lemma 7.3. *Let $\gamma \subset \widetilde{\mathcal{G}}$ be a quasi-geodesic ray, with point at infinity X. If $Q(X) \in \overline{T}$, then $Q(X)$ belongs to the closure of $i(\gamma)$. If $Q(X) \in \partial T$, then $i(\gamma)$ contains a ray going out to $Q(X)$.*

Proof. Fix $\varepsilon > 0$. Choose a basis $\mathcal{A}$ such that $f_{\mathcal{A}} : Z_{\mathcal{A}} \to T$ has backtracking less than ε (see § 1.d). As in [26, proof of 3.1], subdivide $\widetilde{\mathcal{G}}$ to get edges all of simplicial length $< \varepsilon$ and construct an equivariant map $\zeta : \widetilde{\mathcal{G}} \to Z_{\mathcal{A}}$ such that $f_{\mathcal{A}} \circ \zeta$ is 2ε-close to i. As $Z_{\mathcal{A}}$ is a tree, the image of γ in $Z_{\mathcal{A}}$ contains a ray ρ going out to X. If $Q(X) \in \overline{T}$, it is 2ε-close to $\overline{f_{\mathcal{A}}(\rho)}$ by Lemma 1.3, hence 4ε-close to $i(\gamma)$. If $Q(X) \in \partial T$, then $f_{\mathcal{A}}(\rho)$ contains a ray going out to $Q(X)$ by Lemma 1.3. $\square$

Let $BBT(i)$ denote the backtracking constant of $i : \mathcal{G}_{PF} \to T$ (see Lemma 6.2).

Corollary 7.4. *If $\gamma \subset \widetilde{\mathcal{G}}$ is a K-PF-geodesic ray, with point at infinity X, and i maps the origin of γ to $Q(X) \in \overline{T}$, then $i(\gamma)$ is contained in the $BBT(i)$-ball centered at $Q(X)$.*

Proof. Suppose a point $x \in \gamma$ is mapped by i at distance $> BBT(i)$ from $Q(X)$. Since γ is PF-geodesic, the point at distance $BBT(i)$ from $i(x)$ on the segment $[Q(X), i(x)]$ separates $Q(X)$ from $i(y)$ for $y \in \gamma$ closer to X than x. This contradicts Lemma 7.3. $\square$

An inequality.

Recall that $ILT(x, y)$ is the minimum number of illegal turns in any path from x to y in $\widetilde{\mathcal{G}}$.

Lemma 7.5. *There exist constants C_1, C_2 such that*

$$C_1 ILT(x, y) \leq d_{PF}(x, y) \leq (ILT(x, y) + 1)(d_\infty(x, y) + C_2)$$

for all vertices $x, y \in \widetilde{\mathcal{G}}$.

Proof. The first inequality is clear, since an illegal turn involves at least one top edge. For the other, choose a path γ from x to y with minimal number of illegal turns and divide it into legal subpaths γ_j. We define a subset $J \subset \gamma$ in the following way: the intersection of J with γ_j is the maximal subpath $J_j \subset \gamma_j$ bounded by points of the full preimage $r^{-1}([x, y])$ (where $[x, y]$ denotes the segment joining x and y in the tree $\widetilde{G}$). Note that $\gamma \setminus J$ consists of at most $ILT(x, y)$ intervals.

If u, v bound a component of $\gamma \setminus J$, they have the same image in $\widetilde{G}$. Therefore their simplicial distance (hence also their PF-distance) is bounded. We complete the proof by showing that the PF-length of a J_j is bounded by the sum of $d_\infty(x, y)$ and a constant.

Recall that $\pi : \widetilde{G} \to T$ has backtracking bounded by some C', and that i and $\pi \circ r$ are C''-close for some C''. If $u, v \in r^{-1}([x, y])$, then in T we have

$$d_T(\pi r(u), \pi r(v)) \leq d_T(\pi(x), \pi(y)) + 2C'$$

and therefore

$$d_\infty(u, v) \leq d_\infty(x, y) + 2C' + 4C''.$$

This bounds the PF-length of a J_j, because $d_{PF}(u, v) = d_\infty(u, v)$ if u, v are joined by a legal path. $\square$

8. Proof of Theorem 5.1

We fix $\alpha \in \mathrm{Aut}\,(F_k)$, and we assume that there is no simplicial α-invariant tree with trivial arc stabilizers.

Paired train tracks.

We need to consider both a train track G for α and a train track G' for α^{-1}. We want them to be "paired", in particular we want the corresponding trees T and T' to have the same elliptic elements.

Let G be given by [3, 5.1.5]. We may assume that the top stratum G_t is exponentially growing, since otherwise there is a simplicial α-invariant tree (obtained from $\widetilde{G}$ by collapsing all zero edges).

Now apply [3, 5.1.5] to α^{-1} and the free factor system $\mathcal{F} = \mathcal{F}(G \setminus G_t)$ (consisting of fundamental groups of noncontractible components of the zero part of G), obtaining a train track G' for α^{-1}. We may assume that the top stratum of G' is also exponentially growing (otherwise there is a simplicial invariant tree). Since the train track maps given by [3, 5.1.5] are reduced, the free factor systems $\mathcal{F}$ and $\mathcal{F}'$ (associated to the zero parts of G and G') are equal.

All the constructions of § 6 may be applied to G'. We use $'$ to denote the corresponding objects. In particular, we have a graph $\mathcal{G}'$ (obtained from G' by adding a shortcut if needed), a map $f' : \widetilde{\mathcal{G}}' \to \widetilde{\mathcal{G}}'$, a PF metric on $\widetilde{\mathcal{G}}'$, and an $\mathbf{R}$-tree T'.

Since $\mathcal{F} = \mathcal{F}'$, the set of conjugacy classes of F_k represented by loops in the zero part is the same for G and G'. Furthermore, the fundamental groups of noncontractible components of the zero parts of $\mathcal{G}$ and $\mathcal{G}'$ map onto the same subgroups in F_k (this follows from Lemma 6.3 and the remarks following it, noting that α and α^{-1} have the same invariant conjugacy classes). This implies that T, T' have the same elliptic elements, and also:

Lemma 8.1. *The spaces $\widetilde{\mathcal{G}}_{PF}$ and $\widetilde{\mathcal{G}}'_{PF}$ are F_k-equivariantly quasi-isometric.*

Note that, of course, any equivariant map from $\widetilde{\mathcal{G}}$ to $\widetilde{\mathcal{G}}'$ is a quasi-isometry for the *simplicial* metrics.

Proof. This is standard, and we only sketch an argument (compare Proposition 3.1 of [13]). Without changing the quasi-isometry type of $\widetilde{\mathcal{G}}_{PF}$, or images of the zero parts in F_k, we may contract to a point a zero edge of $\mathcal{G}$ with distinct endpoints, or contract a top edge if its endpoints are distinct and at most one touches the zero part.Using these operations and their inverses, we may assume that $\mathcal{G}$ has the following standard form: it has one central vertex v, with top loops θ_i attached, and top edges vv_j with a zero component Z_j attached at each v_j.

The space $\widetilde{\mathcal{G}}_{PF}$ is then quasiisometric to the Cayley graph of F_k with respect to the infinite generating system consisting of (the images in F_k of) the θ_i's and the whole fundamental groups $\pi_1(Z_j, v_j)$. The space $\widetilde{\mathcal{G}}'_{PF}$ has a similar structure, and the two Cayley graphs are quasi-isometric because one can express each element of one generating system as a word of bounded length in the other system. □

Creating a fixed point.

We want f to have a fixed point R in $\widetilde{\mathcal{G}}$. To achieve this, we may have to add an edge to $\mathcal{G}$ (compare [29, § 6]).

If the fixed point Q of H is in $\overline{T} \setminus T$, there is an eigenray ρ (see § 1). Choose $x \in \widetilde{\mathcal{G}}$ with $i(x) \in \rho$. By Lemma 6.1, there is a legal path between $f^p(x)$ and $f^{p+1}(x)$ for p large. Recall that legal paths map PF-isometrically into T. We may assume that $y = f^p(x)$ and all its further images by f are interior points of top edges (this rules out countably many choices for the image of x on ρ).

We now attach an extra edge $[R, y]$ to $\widetilde{\mathcal{G}}$ (or rather an edge $[wR, wy]$ for every $w \in F_k$). We extend f by mapping $[R, y]$ to the segment $[R, f(y)]$ (which contains y), keeping the relation $\alpha(w)f = fw$ satisfied. The new edge is a top edge, with PF-length chosen so that $d_{PF}(R, f(y)) = \lambda d_{PF}(R, y)$. One of the new turns at y is legal, the other is not.

Everything said above extends to this enlarged space, still denoted $\widetilde{\mathcal{G}}$. In particular, Lemmas 6.1 and 6.2 still hold. The associated $\mathbf{R}$-tree (still denoted by T) consists of the previous T together with the orbit of Q. The image of the new edge $[R, y]$ in the tree is the initial segment $[Q, i(y)]$ of the eigenray.

If H has a fixed point in T, lift it to $x \in \widetilde{\mathcal{G}}$ and choose p so that there is a legal path between $y = f^p(x)$ and $f(y) = f^{p+1}(x)$. Since legal paths map PF-isometrically into T, and $i \circ f = H \circ i$, this path contains only zero edges. If $f(y) \neq y$, we attach an edge $[R, y]$ as above, but now the new edge is a zero edge.

The main argument.

We fix $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}'$, paired as above. Adding an edge if needed, we may assume that there are points R, R' in $\widetilde{\mathcal{G}}$, $\widetilde{\mathcal{G}}'$ fixed by f, f'. They project to the fixed points Q, Q' of H and H'. We denote by $\mathcal{C}$, $\mathcal{C}'$ the zero components of $\widetilde{\mathcal{G}}$, $\widetilde{\mathcal{G}}'$ containing R, R'. Recall that $\partial\widetilde{\mathcal{G}}$ and $\partial\widetilde{\mathcal{G}}'$ are identified to ∂F_k.

The stabilizer of $\mathcal{C}$ in F_k is $\operatorname{Stab} Q$ (see § 6). It may be characterized as the only stabilizer which is invariant under α as a subgroup (not just up to conjugacy). In particular, $\operatorname{Stab} Q = \operatorname{Stab} Q'$.

We always assume that the numbers C, K, C_1, C_2 introduced in § 7 work for both $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}}'$. Unless mentioned otherwise, distance refers to the simplicial metric.

We consider $X \in \partial F_k$ with $Q(X) = Q$. The numbers p, μ, ν introduced in the next lemmas will not depend on X. These lemmas apply to the whole $\partial\alpha$-orbit of X since $Q(\partial\alpha^n(X)) = Q$ for every $n \in \mathbf{Z}$.

Lemma 8.2. *Given $\varepsilon > 0$, and K sufficiently large, there exist $p = p(\varepsilon)$ and $\mu = \mu(\varepsilon, K)$ such that*

$$d_{PF}(R, f^p(P)) \leq \varepsilon \ d_{PF}(R, P) + \mu$$

for all points P on a K-quasi-geodesic ray $\gamma = [R, X) \subset \widetilde{\mathcal{G}}$ with $Q(X) = Q$.

Proof. Choose δ with $\delta(BBT(i) + C_2) < C_1\varepsilon$, where C_1, C_2 come from Lemma 7.5. Fix p as in Lemma 6.1 (depending on δ).

Let γ_p be a K-PF-geodesic from R to $\partial\alpha^p(X)$. We claim that $f^p(P)$ is D-close to some $S \in \gamma_p$, with D depending only on K and p (but not on P or X). This is because f^p is a quasi-isometry, so $f^p(\gamma)$ is a quasi-geodesic from R to $\partial\alpha^p(X)$ with quasi-geodesy constants depending only on K and p.

We may assume that D also bounds $ILT(f^p(P), S)$ and $d_{PF}(f^p(P), S)$. Then

$$ILT(R, S) - D \leq ILT(R, f^p(P)) \leq \delta ILT(R, P) \leq \delta/C_1 \, d_{PF}(R, P)$$

by 6.1 and 7.5, and

$$d_{PF}(R, f^p(P)) \leq d_{PF}(R, S) + D \leq (ILT(R, S) + 1)(d_\infty(R, S) + C_2) + D$$

by 7.5.

Note that $d_\infty(R, S)$ is the distance between $Q = i(R)$ and $i(S)$ in T. Since γ_p is a K-PF-geodesic with origin R, and its point at infinity $\partial\alpha^p(X)$ satisfies $Q(\partial\alpha^p(X)) = Q$, Corollary 7.4 yields $d_\infty(R, S) \leq BBT(i)$. We obtain an upper bound for $d_{PF}(R, f^p(P))$, which is a linear function of $d_{PF}(R, P)$ with slope $(\delta/C_1)(BBT(i) + C_2)$. The lemma follows. □

Lemma 8.1 yields an F_k-equivariant map $\varphi : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}'$ which is a quasi-isometry for the PF-metrics. It is also a quasi-isometry for the simplicial metrics. Similarly, choose $\psi : \widetilde{\mathcal{G}}' \to \widetilde{\mathcal{G}}$. Let L be an upper bound for all quasi-isometry constants. Fix ε with $2L^2\varepsilon < 1$ and choose p as in Lemma 8.2.

Lemma 8.3. *Given $K' \geq 0$, there exists $\nu \geq 0$ such that*

$$d_{PF}(R', f'^p(P')) \geq 2d_{PF}(R', P') - \nu$$

for all points P' on a K'-quasi-geodesic ray $\gamma' = [R', X) \subset \widetilde{\mathcal{G}}'$ with $Q(X) = Q \in T$.

Note that the equality $Q(X) = Q$ does take place in T (not in T').

Proof. Like ν, the numbers D_1, D_2, D_3 introduced later in this proof will depend on the choices made above, and on K', but not on P' or X. There exists D_1 such that $\psi(P')$ belongs to a D_1-quasi-geodesic θ from R to X. Choose a K-quasi-geodesic γ from R to $\alpha^{-p}(X)$. Since θ and $f^p(\gamma)$ are quasi-geodesics with the same endpoints, $\psi(P')$ is D_2-close to a point of the form $f^p(P_1)$, with $P_1 \in \gamma$. Finally, we observe that $\varphi(P_1)$ is D_3-close to $f'^p(P')$, because $f'^p \circ \varphi \circ f^p$ is F_k-equivariant, hence at a bounded distance from φ.

Now we consider PF-distance. Working modulo additive constants, we have

$$d_{PF}(R', P') \leq Ld_{PF}(R, f^p(P_1)) \leq L\varepsilon d_{PF}(R, P_1) \leq L^2\varepsilon d_{PF}(R', f'^p(P'))$$

(the second inequality comes from Lemma 8.2, the other two from the quasi-isometry properties of φ and ψ). Lemma 8.3 follows since we have chosen $L^2\varepsilon < \frac{1}{2}$. □

Fix K' so that Lemma 7.2 applies in $\widetilde{\mathcal{G}}'$.

Lemma 8.4. *If $Q(X) = Q$, there exists a legal K'-quasi-geodesic between R' and X in $\widetilde{\mathcal{G}}'$.*

Proof. Let γ' be a K'-PF-geodesic ray between R' and X. Fix $P \in \gamma'$. By 7.1 (and its extension to quasi-geodesic rays), it is C-close to all quasi-geodesics between R' and X. We apply this to $f'^{np}(\gamma'_n)$, where n is a large integer and γ'_n is a K'-PF-geodesic between R' and $\partial\alpha^{np}(X)$. We see that P is C-close to $f'^{np}(P_n)$ for some $P_n \in \gamma'_n$.

By 8.3 applied to γ'_n we have

$$d_{PF}(R', f'^{np}(P_n)) - \nu \geq 2^n(d_{PF}(R', P_n) - \nu),$$

so $d_{PF}(R', P_n) \leq \nu + 1$ for n large since the left hand side is bounded. This gives a uniform bound for $ILT(R', P_n)$, and by 6.1 we deduce $ILT(R', f'^{np}(P_n)) = 0$ for n large.

By 7.2, there exists a legal K'-quasi-geodesic between R' and $f'^{np}(P_n)$. Since $f'^{np}(P_n)$ is C-close to P, we conclude by a diagonal argument, letting P go to infinity on γ'. □

Since $Q(\partial\alpha^n(X)) = Q$, we get:

Corollary 8.5. *If $Q(X) = Q$, then for every $n \in \mathbf{Z}$ there exists a legal K'-quasi-geodesic between R' and $\partial\alpha^n(X)$ in $\widetilde{\mathcal{G}}'$.* □

For notational simplicity we state the next results in $\widetilde{\mathcal{G}}$, even though we will apply them in $\widetilde{\mathcal{G}}'$.

Remark 8.6. If ρ is an eigenray of H (so that $X = j(\rho)$ is a fixed point of $\partial\alpha$), there exists a legal quasi-geodesic γ between R and X in $\widetilde{\mathcal{G}}$ (take $x \in \widetilde{\mathcal{G}}$ mapping into ρ; then $f^n(x) \to X \in \partial\widetilde{\mathcal{G}}$, and for n large there is a legal K-quasi-geodesic between R and $f^n(x)$ by 6.1 and 7.2; in fact, γ consists of an initial segment contained in the zero part, followed by a legal ray in $\widetilde{\mathcal{G}}$). More generally, there is a legal quasi-geodesic between R and wX for $w \in \operatorname{Stab} Q$.

The following lemma is a converse to this remark.

Lemma 8.7. *Let $X \in \partial F_k$. Suppose that for every $n \in \mathbf{N}$ there exists a legal quasi-geodesic ray between R and $\partial\alpha^{-n}(X)$ in $\widetilde{\mathcal{G}}$. If $X \notin \partial\operatorname{Stab} Q$, there exist $q \geq 1$ and $w \in \operatorname{Stab} Q$ such that X is a fixed point of $\partial(i_w \circ \alpha^q)$.*

Proof. Let E, E' be oriented top edges with origin in $\mathcal{C}$, the zero component of $\widetilde{\mathcal{G}}$ containing R. Their images in T are non-degenerate arcs with origin Q. In the special situation that E, E' are in the same F_k-orbit, but distinct, $i(E)$ and $i(E')$ don't overlap (because arc stabilizers of T are trivial). When E, E' are arbitrary (but distinct), we get a positive lower bound for possible overlaps between $i(E)$ and $i(E')$, hence also for possible overlaps of images of legal quasi-geodesics with origin in $\mathcal{C}$.

Let γ be a legal quasi-geodesic from R to X. If it contains only zero edges, then $X \in \partial \mathrm{Stab}\, Q$. From now on we assume that γ has positive PF-length. Its image in T is a non-degenerate (possibly open) segment $s(X)$ with origin Q, which depends only on X (Lemma 7.3 implies that $s(X)$ is the set B_X defined in § 1, possibly with $Q(X)$ added or removed). Note that $s(\partial\alpha(X)) = H(s(X))$.

We claim that there exist $q \geq 1$ and $w \in \mathrm{Stab}\, Q$ such that $s(X)$ and $s(w\partial\alpha^q(X))$ have nontrivial overlap.

There is an action of f_0 on the (finite) set of oriented top edges of $\mathcal{G}$ (associate to each top edge the first top edge of the image edge path), so there exists r such that f_0^r of every element is periodic. Considering the image by f^r of a legal quasi-geodesic from R to $\partial\alpha^{-r}(X)$ in $\widetilde{\mathcal{G}}$, we see that there exists a legal quasi-geodesic γ from R to X such that the initial top edges of γ and some $f^q(\gamma)$ are in the same F_k-orbit (hence in the same $\mathrm{Stab}\, Q$-orbit since $\mathrm{Stab}\, Q$ is the set of deck transformations mapping $\mathcal{C}$ to itself). This proves the claim.

Let $\beta = i_w \circ \alpha^q$, so $w\partial\alpha^q(X) = \partial\beta(X)$. Since $\partial\beta^{-n}(X)$ is in the same $\mathrm{Stab}\, Q$-orbit as $\partial\alpha^{-nq}(X)$, there is a legal quasi-geodesic γ_{-n} from R to $\partial\beta^{-n}(X)$. If the overlap between $s(X)$ and $s(\partial\beta(X))$ is finite, then the overlap between $i(\gamma_{-(n+1)})$ and $i(\gamma_{-n})$ is finite and proportional to λ^{-nq}. This is a contradiction since we have seen that it cannot be arbitrarily small.

It follows that $s(X)$ equals $s(\partial\beta(X))$ and has infinite length, so that it is an eigenray of the homothety wH^q associated to β. We conclude that $X = j(s(X))$ is a fixed point of $\partial\beta$. □

We can now conclude.

Proof of Theorem 5.1. Recall that $\mathrm{Stab}\, Q = \mathrm{Stab}\, Q'$. If $Q(X) = Q$, we may apply 8.7 in $\widetilde{\mathcal{G}}'$ (thanks to 8.5). If $X \notin \partial\mathrm{Stab}\, Q$, we obtain that X is a fixed point of $\partial(i_w \circ \alpha^{-q})$ (the exponent is negative because $\mathcal{G}'$ is a train track for α^{-1}). Of course this implies that X is a fixed point of $\partial(i_{\alpha(w^{-q})} \circ \alpha^q)$, as required. □

9. More on the dynamics

Products of trees.

The techniques used in the previous sections also give:

Theorem 9.1. *Given $\alpha \in \mathrm{Aut}\,(F_k)$, there exist an α-invariant $\mathbf{R}$-tree T and an α^{-1}-invariant $\mathbf{R}$-tree T', as in Theorem 1.2, and there exists $\varepsilon > 0$, such that for every $g \in F_k$ one of the following holds:*

1. *g is elliptic in T and T';*
2. *g is hyperbolic in T and T', and has translation length $> \varepsilon$ in T or in T' (or in both).*

This theorem was proved in [2] and [28] for α irreducible with irreducible powers (no proper free factor of F_k is α-periodic, up to conjugacy). It means that the diagonal action of F_k on $T \times T'$ is discrete. See [18] for further results about such actions.

Proof. The result is clear if there exists a simplicial α-invariant tree T, with $T' = T$. If not, we let T, T' be as in § 8. We use the same notations.

We already know that T and T' have the same elliptic elements. We assume that there is a sequence g_n with $\ell(g_n)$ and $\ell'(g_n)$ positive and going to 0, and we argue towards a contradiction.

If $x, y \in \widetilde{\mathcal{G}}$ satisfy $d_\infty(x, y) \leq 1$ and $ILT(x, y) \geq 1$, then by 7.5 the ratio between $d_{PF}(x, y)$ and $ILT(x, y)$ lies between C^{-1} and C for some $C > 1$. Similar considerations apply to $\mathcal{G}'$, we fix C working for both $\mathcal{G}$ and $\mathcal{G}'$.

Represent (the conjugacy class of) g_n by a PF-geodesic loop $\gamma_n \subset \mathcal{G}$. We write $ILT(\gamma_n)$ for the number of illegal turns of γ_n. If $ILT(\gamma_n)$ remains bounded, then by 6.1 there exists r such that $\alpha^r(g_n)$ is represented by a legal loop. The PF-length of that loop is bounded away from 0, and so is $\ell(g_n) = \lambda^{-r}\ell(\alpha^r(g_n))$ by 6.2. Assume therefore that $ILT(\gamma_n)$ goes to infinity.

Let L be as above (quasi-isometry constant). Fix $\delta > 0$ with $LC^2\delta < 1$. Let p be given by 6.1 (applied to δ in both $\widetilde{\mathcal{G}}$ and $\widetilde{\mathcal{G}'}$). Choose $x \in \widetilde{\mathcal{G}}$ projecting into γ_n in $\mathcal{G}$ and into the axis of g_n in T, and let $y = g_n x$. Note that $d_\infty(x, y) = \ell(g_n)$ and $d_\infty(f^p(x), f^p(y)) = \lambda^p d_\infty(x, y)$ go to 0 as $n \to \infty$.

For n large we have

$$d_{PF}(f^p(x), f^p(y)) \leq C \cdot ILT(f^p(x), f^p(y)) \leq C\delta \cdot ILT(x, y) \leq C^2\delta d_{PF}(x, y),$$

showing that $h_n = \alpha^p(g_n)$ may be represented in $\mathcal{G}$ by a loop of PF-length less than $C^2\delta|\gamma_n|_{PF}$, hence in $\mathcal{G}'$ by a loop of PF-length less than $LC^2\delta|\gamma_n|_{PF} + L$.

Since $\ell'(h_n)$ goes to 0, we may apply the same argument to h_n in $\widetilde{\mathcal{G}'}$, and we get $|\gamma_n|_{PF} \leq (LC^2\delta)^2|\gamma_n|_{PF} + D$ for some constant $D = D(L, C, \delta)$. This shows that $|\gamma_n|_{PF}$, hence $ILT(\gamma_n)$, is bounded, a contradiction. □

Dynamics of irreducible automorphisms.

Consider $\alpha \in \mathrm{Aut}\,(F_k)$. For simplicity assume that all periodic points of $\overline{\alpha}$ are fixed points (this may be achieved by raising α to some power). Recall [15] that fixed points of $\partial\alpha$ not in $\partial\mathrm{Fix}\,\alpha$ are either attracting or repelling, and the action of $\mathrm{Fix}\,\alpha$ on $\mathrm{Fix}\,\partial\alpha \setminus \partial\mathrm{Fix}\,\alpha$ has finitely many orbits.

When $\mathrm{Fix}\,\alpha$ is trivial, $\mathrm{Fix}\,\partial\alpha$ is the vertex set of a *finite bipartite graph* Γ, with an edge from a repelling point X_1 to an attracting point X_2 if and only if there exists $X \in \partial F_k$ with $\lim_{n\to+\infty} \partial\alpha^{-n}(X) = X_1$ and $\lim_{n\to+\infty} \partial\alpha^n(X) = X_2$. Note that every component of Γ contains at least two vertices.

Recall that an automorphism α is irreducible with irreducible powers (iwip) if no proper free factor of F_k is α-periodic (up to conjugacy).

Dynamics of geometric iwip automorphisms (induced by a pseudo-Anosov homeomorphism of a compact surface with one boundary component) is well-understood. Because there is an invariant cyclic ordering on ∂F_k, the graph Γ (defined when $\mathrm{Fix}\,\alpha$ is trivial) is either a single edge or it is homeomorphic to a circle (the second author has conjectured that this property leads to a characterization of geometric automorphisms).

We focus on non-geometric iwip automorphisms (in this case, $\mathrm{Fix}\,\alpha$ is always trivial [6]). We do not know which graphs Γ may appear in this context. We only prove:

Theorem 9.2. *Let $\alpha \in \mathrm{Aut}\,(F_k)$ be irreducible with irreducible powers, not geometric. Assume that all periodic points of $\partial\alpha$ are fixed points. Then either the graph Γ has exactly two vertices, or every component of Γ contains strictly more than two vertices.*

Proof. Let T be an invariant $\mathbf{R}$-tree as in Theorem 1.2 (it is unique up to rescaling in this case, see [26]). It is well-known that $\mathrm{Fix}\,\alpha$ is trivial, and the action of F_k on T is free (this may be deduced from [6] and Lemma 6.3). We have $\lambda > 1$, and we let Q be the fixed point of H (in $\overline{T}$).

By Proposition 3.4, all components of $\overline{T} \setminus \{Q\}$ are fixed by H. If $g \in F_k$ is nontrivial, then $\alpha^n(g)$ converges to the attracting fixed point $j(\rho)$, where ρ is the eigenray of H contained in the component containing gQ (see the proof of Theorem 3.1). There is a similar relation between the α^{-1}-invariant tree T' and repelling fixed points of $\partial\alpha$.

We may assume that $\partial\alpha$ has at least two attracting fixed points, and two repelling ones (otherwise, the result is trivial). By 3.4 this implies that $\overline{T} \setminus \{Q\}$ has at least two components, so Q belongs to T (not just to $\overline{T}$). Similarly $Q' \in T'$. We now argue by way of contradiction, assuming that some component of Γ has only two vertices, an attracting fixed point X and a repelling point X'. Let $\mathcal{C}, \mathcal{C}'$ be the corresponding components of $T \setminus \{Q\}$, $T' \setminus \{Q'\}$ respectively, and ρ the eigenray of H contained in $\mathcal{C}$.

Viewing nontrivial elements of F_k as hyperbolic isometries of T, we claim that *there exists $g \in F_k$ whose translation axis A_g passes through Q and intersects ρ in a segment strictly longer than the translation length $\ell(g)$ (i.e. gQ is an interior point of $A_g \cap \rho$).*

Assuming this claim temporarily, we complete the proof as follows. The point $g^{-1}Q$ belongs to a component $\mathcal{C}_1$ of $T \setminus \{Q\}$ distinct from $\mathcal{C}$. Since the edge $X'X$ of Γ is isolated, the point $g^{-1}Q' \in T'$ belongs to a component $\mathcal{C}'_1$ of $T' \setminus \{Q'\}$ distinct from $\mathcal{C}'$ (otherwise Γ would contain an edge between X' and the attracting fixed point corresponding to $\mathcal{C}_1$). Now consider $g^{-1}\alpha^p(g)$, for p large. The point $g^{-1}\alpha^p(g)Q = g^{-1}H^p(gQ) \in T$ belongs to $\mathcal{C}$ because of our choice of g. In T', on the other hand, the point $g^{-1}\alpha^p(g)Q' = g^{-1}(H')^{-p}(gQ')$ is close to $g^{-1}Q'$ and therefore belongs to $\mathcal{C}'_1$. It follows that X is joined to two distinct vertices of Γ, a contradiction.

There remains to prove the claim. We use the terminology of § 6. Since $\partial\alpha$ has at least two attracting fixed points, it follows from [15, p. 431] that f has a fixed point $R \in \widetilde{\mathcal{G}}$. By Remark 8.6, there exists a legal quasi-geodesic γ between R and $X = j(\rho)$. By irreducibility of α, the projection of γ onto $\mathcal{G}$ passes again over its initial top edge (with the same orientation). This defines a loop in $\mathcal{G}$, and a nontrivial $g \in F_k$ mapping an initial segment of γ into γ (in an orientation-preserving way). This is the required g. ◻

The number of periods.

Given $\alpha \in \mathrm{Aut}\,(F_k)$, recall [25] that periods of elements of F_k for α are bounded by A_k, the maximum order of torsion elements in $\mathrm{Aut}\,(F_k)$, and periods of elements of ∂F_k are bounded by $M_k = 2kA_k$. For k large, one has $\log A_k \sim \log M_k \sim \sqrt{k \log k}$.

Examples of automorphisms with many periods may be constructed as follows. Let p be a prime number. Let σ be a permutation consisting of one cycle of order p' for each prime number $p' \leq p$. It defines a periodic automorphism α of F_k, with $k = 2+3+\cdots+p$. The periods of $\overline{\alpha}$ are exactly the divisors of $2 \cdot 3 \cdots p$. A simple computation (based on the prime number theorem) shows that, for p large, the logarithm of the number of periods of $\overline{\alpha}$ is asymptotic to $2\log 2 \cdot \sqrt{\frac{k}{\log k}}$.

Theorem 9.3. *Given $\alpha \in \mathrm{Aut}\,(F_k)$, the number of periods of periodic points of $\overline{\alpha}$ is bounded by N_k, with $\log N_k \sim 2\log 2 \cdot \sqrt{\frac{k}{\log k}}$.*

Proof. First consider periods of elements of F_k. Let $P(\alpha)$ be the periodic subgroup (consisting of all α-periodic $g \in F_k$). Since $P(\alpha)$ is the fixed subgroup of some power of α, it has rank at most k by [6]. The number of periods of α is bounded by the number of divisors of $o(\alpha)$, where $o(\alpha)$ is the order of α in $\mathrm{Aut}\,(P(\alpha))$. Now $o(\alpha) \leq A_k$, and the number $d(n)$ of divisors of n satisfies $\limsup_{n\to+\infty} \dfrac{\log d(n) \log\log n}{\log n} = \log 2$ (see [19, Theorem 317]). Replacing n by A_k with $\log A_k \sim \sqrt{k \log k}$ gives N_k as in the theorem.

If $X \in \partial F_k$ is periodic, and no $g \in F_k$ has the same period, the proof of Theorem 2.1 of [25] shows that the period of X divides rs, where $r \leq 2k$ and s is the period of some $g \in F_k$. The estimate therefore also holds for the periods of $\partial\alpha$ since $\log 2kN_k \sim \log N_k$. $\square$

Automorphisms with many fixed points.

We give a short proof of a result due to Bestvina, Feighn, Handel [5], improving their lower bound from 3 to 4.

Proposition 9.4. *For any outer automorphism Φ of F_k, $k \geq 2$, there exist $q \geq 1$ and $\beta \in \mathrm{Aut}\,(F_k)$ representing Φ^q such that $\partial\beta$ has at least four fixed points. If $u \in F_k$ is fixed by some $\alpha \in \mathrm{Aut}\,(F_k)$ representing Φ, we may require $\beta(u) = u$.*

Remark. With the terminology of [15, § 6], we shall prove that any $\Phi \in \mathrm{Out}\,F_k$ has a power with positive index. It is not always possible to take $q = 1$: if $\alpha \in \Phi$ cyclically permutes the elements of a free basis of F_k, it follows from [8] that q has to be divisible by k.

Proof. Starting with $\alpha \in \Phi$, we will keep replacing it by a power α^s, or by $i_w \circ \alpha$, so as to finally obtain an automorphism (still denoted by α) with at least four fixed points on ∂F_k.

Let T be an α-invariant $\mathbf{R}$-tree as in Theorem 1.2. Replacing α by α^s or by $i_w \circ \alpha$ amounts to replacing H by H^s or wH.

By assertion (d) of Theorem 1.2, we may assume that H fixes a branch point $Q \in T$ and acts trivially on the set of $\operatorname{Stab} Q$-orbits in $\pi_0(T \setminus \{Q\})$. We distinguish several cases.

If $\lambda > 1$ and $\operatorname{Stab} Q$ is trivial, then $\partial\alpha$ has at least three attracting fixed points (see Proposition 3.4). There is a fourth, repelling, point. If $\lambda > 1$ and $\operatorname{Stab} Q = \mathbf{Z}$, some wH with $w \in \operatorname{Stab} Q$ has an eigenray and there are infinitely many attracting periodic points (because $\operatorname{Fix} \partial\alpha^2$ is invariant under the action of $\operatorname{Stab} Q$). If $\lambda > 1$ and $\operatorname{Stab} Q$ has rank ≥ 2, we use induction on k.

If $\lambda = 1$, we may assume that H fixes an edge $e = [Q, R]$, and we collapse every edge not in the orbit of e. In the new tree, $\operatorname{Stab} Q$ and $\operatorname{Stab} R$ are non-trivial and α-invariant. This proves the first part of the proposition.

Now assume α fixes some nontrivial $u \in F_k$ (and therefore $\partial\alpha$ has two fixed points $u^{\pm\infty}$). Note that H commutes with u. If u fixes a (unique) point $Q \in T$ (in particular if $\lambda > 1$), this point is also fixed by H and we argue as before. Finally, suppose $\lambda = 1$ and u is hyperbolic. In this case we may assume that H equals the identity on the axis A of u (replace H by some $u^r H^s$). After possibly collapsing we get $Q \in A$ with nontrivial stabilizer and we consider $\alpha_{|\operatorname{Stab} Q}$. $\square$

10. Hyperbolic groups

Bounding periods.

Let Γ be a torsion-free hyperbolic group. Given $\alpha \in \operatorname{Aut}(\Gamma)$, let $\operatorname{Fix} \alpha$ be its fixed subgroup, and $P(\alpha) = \cup_n \operatorname{Fix} \alpha^n$ be its periodic subgroup. A subgroup of Γ is a *fixed subgroup* (resp. a *periodic subgroup*) if it equals $\operatorname{Fix} \alpha$ (resp. $P(\alpha)$) for some $\alpha \in \operatorname{Aut}(\Gamma)$.

Proposition 10.1. *Every periodic subgroup is hyperbolic. Up to isomorphism, there are only finitely many periodic subgroups in a given Γ.*

Proof. Most arguments come from [35]. There are two cases.

• Suppose Γ is one-ended (= freely indecomposable). By [34, Theorems 3.2 and 4.1], the group $P(\alpha)$, if not trivial or cyclic, is a vertex group in some splitting of Γ with cyclic edge groups. By [17], such a splitting is obtained from the JSJ splitting constructed in [7] by blowing up vertices with group equal to $\mathbf{Z}$, blowing up surface vertices along disjoint simple closed curves, and then collapsing edges. Finiteness of the set of curves on a compact surface (up to homeomorphism) implies that there are only finitely many possibilities for $P(\alpha)$ (up to an automorphism of Γ). Furthermore, $P(\alpha)$ is hyperbolic because it is a vertex group in a splitting with quasiconvex edge groups [7].

• Now suppose that Γ is the free product of cyclic groups and one-ended groups Γ_i. By the Kurosh subgroup theorem, $P(\alpha)$ is the free product of cyclic groups and subgroups H_j of conjugates K_j of the Γ_i's. The number of factors is the *Kurosh rank* of $P(\alpha)$. It is finite because $P(\alpha)$ is the increasing uni on of the $\operatorname{Fix}(\alpha^{n!})$, whose Kurosh rank is uniformly bounded [10].

If $\alpha^p(\Gamma_i)$ meets Γ_i non-trivially, then $\alpha^p(\Gamma_i) = \Gamma_i$. It follows that each K_j is α-periodic, and therefore $H_j = P(\alpha) \cap K_j$ is the periodic subgroup of some automorphism of K_j. By the first case, H_j is hyperbolic and has only finitely many possibilities (up to isomorphism). The same is therefore true of $P(\alpha)$. $\square$

Since $P(\alpha)$ is finitely generated, the restriction of α to $P(\alpha)$ has finite order. In particular, every periodic subgroup is a fixed subgroup and is quasiconvex [31].

Recall [23] that for P torsion-free hyperbolic there are only finitely many conjugacy classes of torsion elements in $\mathrm{Aut}\,(P)$. Conjugate automorphisms having isomorphic fixed subgroups, we deduce:

Corollary 10.2 (Shor [35]). *Up to isomorphism, there are only finitely many fixed subgroups in a given torsion-free hyperbolic group* Γ. $\square$

We have also proved part of assertion (1) of Theorem III:

Corollary 10.3. *Given a torsion-free hyperbolic group* Γ*, there exists* M *such that, if* $g \in \Gamma$ *is periodic under* $\alpha \in \mathrm{Aut}\,(\Gamma)$*, then the period of* g *is at most* M*.*

Proof. The period of g divides the order of α in $\mathrm{Aut}\,(P(\alpha))$. $\square$

One-ended groups.

Theorem III will first be proved for torsion-free groups, in this subsection (one-ended groups) and in the next (groups with infinitely many ends). Groups with torsion will then be considered.

Let Γ be a torsion-free one-ended hyperbolic group, and $\alpha \in \mathrm{Aut}\,(\Gamma)$. Using Corollary 10.3, we shall prove that *every* $X \in \partial\Gamma$ *is asymptotically periodic*, with a uniform bound (depending only on Γ) for the period of the limiting orbit. This will prove assertions (1) and (3) of Theorem III for Γ. Assertion (2) is proved by similar arguments, or deduced from (3) since the sequences $\alpha^n(g)$ and $\partial\alpha^n(g^\infty)$ have the same limit points when g is not α-periodic (see [24, proof of 2.3]).

We use the JSJ-splitting of Γ, first introduced by Sela. We prefer to follow Bowditch's approach [7] because of its strong uniqueness properties. The JSJ-splitting decomposes Γ as the fundamental group of a finite graph of groups, with associated Bass-Serre tree T. Since T is constructed purely from the topology of $\partial\Gamma$, the group $\mathrm{Aut}\,(\Gamma)$ acts on T in the same way as on $\partial\Gamma$. In particular, there is an isometry $H : T \to T$ as in Theorem 1.2.

Edge stabilizers are cyclic. Vertex stabilizers $\mathrm{Stab}\,Q$ are quasiconvex [7], hence hyperbolic. Furthermore the boundary of Γ is the disjoint union of the set of ends of T (embedded into $\partial\Gamma$ by a map j as in § 1.d) and the (non-disjoint) union of the boundaries $\partial\mathrm{Stab}\,Q$ of the vertex stabilizers (see [7]).

A vertex stabilizer $\mathrm{Stab}\,Q$ is cyclic, or free ("hanging fuchsian"), or "relatively rigid" (the subgroup of $\mathrm{Out}\,(\mathrm{Stab}\,Q)$ consisting of outer automorphisms fixing stabilizers of edges incident to Q (up to conjugacy) is finite; as explained in [23], this follows from [1] and [32]).

If an edge e of T is fixed by H, its stabilizer is an α-invariant cyclic subgroup of Γ. By Corollary 10.3, we may raise α to a fixed power (depending only on Γ) to ensure that any H-periodic edge is in fact fixed.

Suppose that a vertex $Q \in T$ is fixed by H. Then $\mathrm{Stab}\,Q$ is α-invariant, and the induced automorphism has finite order in $\mathrm{Out}\,(\mathrm{Stab}\,Q)$ (if $\mathrm{Stab}\,Q$ is cyclic or relatively rigid) or is a geometric automorphism of a free group (if $\operatorname{Stab} Q$ is hanging fuchsian). It follows that any $X \in \partial(\operatorname{Stab} Q)$ is asymptotically periodic, with a uniform bound on the period.

The proof for arbitrary X now is fairly similar to the proof of Theorem 4.1. Lemma 3.2 extends to actions of torsion-free hyperbolic groups with trivial or cyclic edge stabilizers (compare [11, 21]). Lemma 2.1 also extends to torsion-free hyperbolic groups.

Let $X \in \partial\Gamma$. If the isometry $H : T \to T$ is hyperbolic, with axis A, then either $X = j(A^-)$ is fixed by $\partial\alpha$, or $\partial\alpha^n(X)$ converges to $j(A^+)$ as $n \to +\infty$. If H is elliptic, let $\mathcal{P}$ be its fixed subtree (equal to the set of periodic points).

First suppose $X \in \partial\operatorname{Stab} R$ for some vertex R. Using remarks made above, we may assume that R cannot be chosen in $\mathcal{P}$. Let Q be the point of $\mathcal{P}$ closest to R. Let $\mathcal{C}$ be the component of $T \setminus \{Q\}$ containing R. If $g_p \in \operatorname{Stab} R$ converges to X, we have $g_p Q \in \mathcal{C}$ for p large, because otherwise $g_p \in \operatorname{Stab} Q$ and we could have chosen R in $\mathcal{P}$. Now apply Lemmas 3.2 and 2.1 as in the proof of Theorem 4.1. Note that Remark 2.2 provides a uniform bound for the cardinality of $\omega(X)$.

If $X = j(\rho)$ for some end ρ of T, then either ρ is an end of $\mathcal{P}$ (and X is fixed by $\partial\alpha$), or we apply Lemmas 3.2 and 2.1 using the point $Q \in \mathcal{P}$ closest to ρ and the component $\mathcal{C}$ of $T \setminus \{Q\}$ containing ρ.

Free products.

Let Γ be torsion-free, with infinitely many ends. We write $\Gamma = F_k * \Gamma_1 * \cdots * \Gamma_m$, with each Γ_i one-ended. We will use the invariant $\mathbf{R}$-tree given by the following result.

Theorem 10.4. *Given $\alpha \in \operatorname{Aut}(\Gamma)$, there exist an $\mathbf{R}$-tree T and a homothety H satisfying conditions (a), (b), (c) of Theorem 1.2. Furthermore each Γ_i ($1 \leq i \leq m$) fixes a point of T, and there exists a Γ-equivariant injection $j : \partial T \to \partial\Gamma$ satisfying $\partial\alpha \circ j = j \circ H$.*

Proof. The construction of T in the case of F_k has been sketched in § 6 (see [15] for details): equip $\widetilde{G}$ with the PF-metric d_{PF}, and consider the metric space associated to d_∞ (when $\lambda = 1$, simply collapse components of the zero set of $\widetilde{G}$ to points). The proof in the general case is similar, using the "efficient representatives" constructed by Collins-Turner in [10]. The only difference is that the zero set is a 2-complex (not necessarily a graph), but this difference is irrelevant as each component gets collapsed to a point.

Conditions (a) and (b) of Theorem 1.2 are proved as in [15] (Lemmas 2.7 and 2.8), and Γ_i fixes a point in T because it fixes a component of the zero set in $\widetilde{G}$. The map j is constructed as in [15, Lemmas 3.4 and 3.5], but we need to show that equivariant maps from Cayley graphs of Γ to T have bounded backtracking (in the sense of § 1.d).

This is true for maps to $\widetilde{G}$ (equipped with a simplicial metric), because $\widetilde{G}$ is quasi-isometric to Γ. Therefore it is also true for maps to the tree T_0 obtained from $(\widetilde{G}, d_{PF})$ by collapsing the zero set. When $\lambda > 1$, we further observe that the map

$\overline{f} : T_0 \to T_0$ induced by f is λ-Lipschitz for the PF-metric, and has backtracking bounded by some constant K. Letting $d_p(x, y) = \lambda^{-p} d_{PF}(\overline{f}^p(x), \overline{f}^p(y))$, we deduce that the identity map from (T_0, d_{PF}) to (T_0, d_p) has backtracking bounded by $K(\lambda^{-1} + \cdots + \lambda^{-p})$. The canonical map from (T_0, d_{PF}) to $T = (T_0, d_\infty)$ is 1-Lipschitz, and has bounded backtracking because the series $\lambda^{-1} + \cdots + \lambda^{-p} + \ldots$ converges. It follows that equivariant maps from Cayley graphs of Γ to T have bounded backtracking. □

Recall that the *rank* $\mathrm{rk}\,(J)$ of a group J is the minimum cardinality of a generating set (not to be confused with the Kurosh rank used in [10]).

Theorem 10.5 (Gaboriau [14]). *Let Γ and T be as above. For any $Q \in T$, the stabilizer $\mathrm{Stab}\, Q$ has rank $\mathrm{rk}\,(\mathrm{Stab}\, Q) \leq \mathrm{rk}\,(\Gamma) - 1$, and the action of $\mathrm{Stab}\, Q$ on $\pi_0(T \setminus \{Q\})$ has at most $2\mathrm{rk}\,(\Gamma)$ orbits.* □

We can now prove Theorem III for Γ. Let T be as in Theorem 10.4. Note that for $Q \in T$ the intersection of $\mathrm{Stab}\, Q$ with a conjugate $g\Gamma_i g^{-1}$ is either trivial or the whole of $g\Gamma_i g^{-1}$ (because arc stabilizers are trivial and Γ_i fixes a point). Thus vertex stabilizers are free products, with each factor free or isomorphic to some Γ_i. They are "simpler" than Γ by Theorem 10.5, quasiconvex (see e.g. [21, 36]), and up to isomorphism they belong to some finite set depending only on Γ.

The proof of assertion (2) of Theorem III now proceeds exactly as in the case of F_k (Theorem 3.1), by induction on rank, since the result is already known for one-ended groups.

To prove assertion (1), we need to bound the period of a $\partial\alpha$-periodic $X \in \partial\Gamma$. If $X \in \partial P(\alpha)$, the period is bounded by the order of α in $\mathrm{Aut}\,(P(\alpha))$. If not, then X is attracting or repelling, hence belongs to the ω-limit set (for α or α^{-1}) of every $g \in \Gamma$ close enough to X in $\overline{\Gamma}$ (see the discussion in § 4 of [25]). We therefore reduce to controlling the cardinality of $\omega(g)$.

The arguments from the proof of Theorem 3.1 do not provide uniform bounds, for two reasons. If $\lambda > 1$ and the component $\mathcal{C}$ is H-periodic, we do not have a bound for the period. If $\lambda = 1$ and the isometry H is elliptic, we do not have a bound for the period of its periodic points.

We first show that, *if Q is a fixed point of H, there is a bound depending only on Γ for the period p of an H-periodic component $\mathcal{C}$ of $T \setminus \{Q\}$.* By Theorem 10.5 we may assume that H acts trivially on the set of orbits of the action of $\mathrm{Stab}\, Q$ on $\pi_0(T \setminus \{Q\})$. We then have $H\mathcal{C} = w\mathcal{C}$ for some $w \in \mathrm{Stab}\, Q$, and $H^p\mathcal{C} = w_p\mathcal{C}$ with $w_p = \alpha^{p-1}(w) \ldots \alpha(w)w$. From $w_p = 1$ we get $\alpha^p(w) = w$, hence $\alpha^r(w) = w$ for some r depending only on Γ by Corollary 10.3, and finally $w_r = 1$ because $(w_r)^p = w_{pr} = 1$. This implies $p \leq r$.

Recall that we want to bound the period of $X \in \omega(g)$. If $\lambda > 1$, the arguments from the proof of Theorem 3.1 show that either $X = j(\rho)$ for some H-periodic ray, and we are done, or $\omega(g) \subset \partial\mathrm{Stab}\, Q$ and we can use induction on the rank of Γ.

Now suppose $\lambda = 1$. There is a problem only if H is elliptic and has periodic points with large periods. It then has a fixed point Q, and using the fact proved above about periodic components of $T \setminus \{Q\}$ we may assume that H fixes an edge

e (this involves raising α and H to some power depending only on Γ). Consider the tree T' obtained by collapsing all edges of T not in the Γ-orbit of e.

It satisfies the conditions of Theorem 10.4, and the quotient graph T'/Γ has exactly one edge (it is a segment or a loop). We can now complete the proof by induction, using the special form of α as in [15, pp. 442-443] (if for instance T'/Γ is a segment, then α preserves a nontrivial decomposition of Γ as a free product, and any attracting periodic point X may be written $X = gX'$ where $g \in \Gamma$ is α-periodic and X' is contained in the boundary of a free factor). This completes the proof of assertion (1) of Theorem III.

The proof of assertion (3) is the same as for free groups (proof of Theorem 4.1). Lemma 4.3 extends, replacing M_k by the bound obtained above. This completes the proof of Theorem III for torsion-free groups.

Groups with torsion.

The goal of this subsection is to extend Theorem III to virtually torsion-free groups. We start with general considerations.

Suppose Δ has finite index in a group Γ. Then there exists k such that, for every $\alpha \in \mathrm{Aut}\,(\Gamma)$, every coset of Γ modulo Δ is mapped to itself by α^k. In particular, $\alpha^k(\Delta) = \Delta$ for every $\alpha \in \mathrm{Aut}\,(\Gamma)$. Of course ${\alpha^k}_{|\Delta}$ is polynomially growing if α is.

Suppose furthermore that there is a bound for the order of torsion in Δ. Then *there is a bound for periods of elements of Γ under automorphisms of Γ if there is one for periods of elements of Δ under automorphisms of Δ.* To prove this, suppose $\alpha^p(g) = g$ with $g \in \Gamma$ and $\alpha \in \mathrm{Aut}\,(\Gamma)$. Replacing α by α^k, we may assume $\alpha(g) = wg$ with $w \in \Delta$. Then $\alpha^p(g) = w_p g$ with $w_p = \alpha^{p-1}(w)\dots\alpha(w)w$. As in the previous section, $w_p = 1$ implies $\alpha^p(w) = w$, and therefore $\alpha^r(w) = w$ for some r which can be bounded in terms of Γ and Δ only. We then write $(w_r)^p = w_{pr} = 1$, and we bound the period of g by r times the order of w_r.

Now suppose that Γ is hyperbolic and Δ is a torsion-free subgroup of finite index. We have just proved assertion (1) of Theorem III for periodic orbits of α. Since Δ and Γ have the same boundary, assertions (1) (for orbits of $\partial\alpha$) and (3) hold.

To prove assertion (2), consider $g \in \Gamma$ and $\alpha \in \mathrm{Aut}\,(\Gamma)$. Replacing α by α^k, we may assume $\alpha(\Delta) = \Delta$ and $\alpha(g) = hg$ with $h \in \Delta$. Then $\alpha^n(g) = h_n g$ with $h_n = \alpha^{n-1}(h)\dots\alpha(h)h$, and we conclude by Corollary 3.3 (proved in Δ just like in F_k).

We also show:

Proposition 10.6. *If $\alpha \in \mathrm{Aut}\,(\Gamma)$, with Γ an infinite, virtually torsion-free, hyperbolic group, then $\partial\alpha$ has at least two periodic points. If $\partial\alpha$ has only one periodic orbit, then this orbit is the boundary of an α-invariant virtually cyclic subgroup.*

Proof. As in [25, proof of 1.1]. If $P(\alpha)$ is finite, assertion (2) of Theorem III provides both an attracting periodic orbit and a repelling one. If $P(\alpha)$ is virtually $\mathbf{Z}$, its boundary gives two fixed points, or an orbit of order 2. If $P(\alpha)$ is non-elementary, there are uncountably many periodic orbits. □

11. Examples and questions

Examples.

Fixed points of $\partial\alpha$ not in $\partial\mathrm{Fix}\,\alpha$ are either attracting or repelling. Now consider the automorphism α of F_2 mapping a to a and b to aba. The group $\mathrm{Fix}\,\alpha = \langle a\rangle$ is infinite cyclic, and its two limit points $a^{\pm\infty}$ are isolated fixed points of $\partial\alpha$ which are *half-attracting and half-repelling:* if $X \in \partial F_2$ is not $a^{\pm\infty}$, then the limit of $\partial\alpha^n(X)$ as $n \to +\infty$ is either a^∞ or $a^{-\infty}$, depending on whether the first occurrence of $b^{\pm 1}$ in X is b or b^{-1}. The automorphism α is the square of $\beta : a \mapsto a^{-1}, b \mapsto a^{-1}b^{-1}$. The set $\{a^\infty, a^{-\infty}\}$ is a half-attracting, half-repelling orbit of period two of $\partial\beta$.

The following example of a *parabolic orbit* is due to A. Hilion [20]. Define γ on F_4 by $a \mapsto a$, $b \mapsto ba$, $c \mapsto ca^2$, $d \mapsto dca$. For $g = bad^{-1}$, both sequences $\gamma^n(g)$ and $\gamma^{-n}(g)$ converge to $ba^{-\infty}$ as $n \to +\infty$.

When $\mathrm{Fix}\,\alpha$ is cyclic, it may happen that its limit points are not isolated as fixed points of $\partial\alpha$. Consider a homeomorphism h of a compact surface Σ fixing a separating simple closed curve C pointwise. Assume that h induces a pseudo-Anosov homeomorphism on each of the complementary subsurfaces Σ_ℓ and Σ_r, and h twists non-trivially around C in Σ_ℓ (but not in Σ_r). Let α be the automorphism induced on $\pi_1(\Sigma)$ (with basepoint on C), and $\partial\alpha$ the homeomorphism induced on its boundary. Note that the boundary is a circle (if Σ is closed) or a cyclically ordered Cantor set (if Σ has a boundary). The map $\partial\alpha$ has two fixed points associated to the invariant cyclic subgroup $\pi_1(C)$. They divide the boundary into two intervals I_ℓ and I_r. On I_ℓ, the map $\partial\alpha$ has no fixed point (because of the non-trivial twist). On I_r, there are infinitely many attracting fixed points, and infinitely many repelling ones (but only finitely many orbits under the action of $\pi_1(C)$; they correspond to singular leaves of the invariant foliations of $h_{|\Sigma_r}$ issuing from singularities belonging to C). They alternate on I_r, and accumulate onto both endpoints of I_r.

Free groups.

Consider $\alpha \in \mathrm{Aut}\,(F_k)$. For simplicity we assume in this discussion that all periodic points of $\overline{\alpha}$ are fixed points.

Theorem II asserts that, as $n \to +\infty$, the sequence $\partial\alpha^n(X)$ converges to some $h_\alpha(X) \in \mathrm{Fix}\,\partial\alpha \subset \partial F_k$ for every $X \in \partial F_k$. Let U be the open set $\partial F_k - \mathrm{Fix}\,\partial\alpha$. Is the function h_α locally constant on U? Is the convergence of $\partial\alpha^n$ to h_α locally uniform on U?

Elements of $h_\alpha(U)$ not in $\partial\mathrm{Fix}\,\alpha$ are attracting fixed points. The action of $\mathrm{Fix}\,\alpha$ on the set of attracting fixed points has at most $2k$ orbits [15]. It is proved in [20] that the action of $\mathrm{Fix}\,\alpha$ on $h_\alpha(U) \cap \partial\mathrm{Fix}\,\alpha$ also has finitely many orbits.

Hyperbolic groups.

As shown in § 10, some of our results about automorphisms of free groups may be extended to hyperbolic groups. Another example is Proposition 9.4, which readily extends to non-elementary, virtually torsion-free, hyperbolic groups. On the other hand, we do not know how to prove that exponentially growing automorphisms of an infinitely-ended hyperbolic group Γ have asymptotically periodic dynamics on $\partial\Gamma$.

Let $\alpha \in \mathrm{Aut}\,(\Gamma)$, with Γ hyperbolic. Is there a bound depending only on Γ for the number of $(\mathrm{Fix}\,\alpha)$-orbits of attracting fixed points of $\partial\alpha$? Can one associate a set of growth rates $\Lambda(\Phi) \subset (1, +\infty)$ to $\Phi \in \mathrm{Out}\,(\Gamma)$ as in [25]? As when $\Gamma = F_k$, elements $\lambda \in \Lambda(\Phi)$ should be the growth rates of conjugacy classes under iteration of Φ, and also the rates of convergence towards fixed points with respect to the canonical Hölder structure on $\partial\Gamma$ (see [25]). They should be either dilation factors of pseudo-Anosov homeomorphisms associated to hanging Fuchsian subgroups, or eigenvalues of a transition matrix associated to a (relative) train track coming from a free product structure. In particular, there should be an upper bound to the cardinality of $\Lambda(\Phi)$ that only depends on Γ (not on Φ).

Actions with finite limit sets.

There are many examples of actions of a group Γ on a compact space X with the following property: *There exists $q \geq 1$ such that, for every $x \in X$ and $g \in \Gamma$, the sequence $g^{qn}(x)$ converges as $n \to +\infty$.* For instance:

• The action of $\mathrm{Aut}\,(F_k)$ on ∂F_k and $\overline{F}_k$. Conjecturally, the action of $\mathrm{Aut}\,(\Gamma)$ on $\overline{\Gamma}$, for an arbitrary hyperbolic group Γ.

• Convergence actions of virtually torsion-free groups.

• The action of the mapping class group of a closed surface on the Thurston boundary of Teichmüller space (this follows from Nielsen-Thurston theory). By analogy, one may ask about the action of $\mathrm{Out}\,(F_k)$ on the boundary of Culler-Vogtmann's outer space (see [4], [9], [26] for partial results).

• The action of $\pi_1 M$ on the sphere at infinity of $\widetilde{M}$, where M is a closed Riemannian manifold (or orbifold) with negative curvature and $\widetilde{M}$ is the universal covering. Flat manifolds also provide examples, because of Bieberbach's theorem. To what extent may this be extended to arbitrary non-positively curved manifolds, or even to arbitrary CAT(0) spaces?

Knowing that a group Γ acts on X with the above property gives a lot of information about dynamics of individual elements of Γ. The only global information, however, is the fact that q does not depend on g. There may exist a stronger property, that would be weaker than the convergence property but strong enough to contain more global information on Γ.

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